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STABILITY ANALYSIS OF A HIGH PRESSURE

PNEUMATIC MECHANICAL SYSTEM

A THESIS

Presented to

the Faculty of the Graduate Division

by

William Hoyt Ford

In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Engineering Mechanics

Georgia Institute of Technology

June, 1962

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Approved:

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Date Approved by Chairman: May 30, 1962

ACKNOWLEDGMENTS

The basic concept of this report was evolved while the author was attached to the Mechanical and Hydraulic Staff, Dept. 72-42, Lockheed Aircraft Corporation, Georgia Division. He extends his thanks to that group for supplying him with the analysis portion of the original problem entitled "Design, Development and Functional Test of a 1000°F Pneumatic System, Phase II," because it was during this particular program that the idea occurred to him to expand Evan's Root Locus Method from his single parameter study to an n parameter study.

The author thanks Dr. C. E. Stoneking, his thesis advisor, and Professors F. M. Hill, Dr. Robin B. Gray, and Karl M. Murphy, for their work and suggestions which made the report more understandable.

Finally, the author expresses his gratitude to his wife, Jean, for her untiring mental and physical prodding which resulted in his continuing with the work until it was finished.

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LIST OF SYMBOLS

SYMBOL	DEFINITION	UNITS
A	Piston head area	in^2
C_c	Capillary resistance coefficient	$in^5/lb\text{-}sec$
C_L	Velocity feedback damping term	lb-sec/in
F_b	Reaction force on feedback connecting rod	lb
F_s	Force at end of damper arm	lb
F_v	Reaction force on valve arm due to valve coulomb friction	lb
g	Acceleration due to gravity	in/sec^2
a,b,c,d,e, f,g, & h	Linkage lengths (see Figure 1)	in
k	Specific heat ratio of air = 1.4	--
k_1	Valve coefficient denoting no-load flow sensitivity	in^2/sec
k_2	Valve coefficient denoting loss of flow per unit of load pressure	$in^5/lb\text{-}sec$
k_3	Coefficient denoting fluid compliance	in^5/lb
K_L	Air load spring rate	lb/in
K_s	Velocity feedback spring rate	lb/in
L_c	Capillary passage length	in
m_L	Effective mass displaced by the actuator	$lb\text{-}sec^2/in$
P_i	Initial pressure	lb/in^2
P_m	Pressure difference across piston	lb/in^2
R	Proportional feedback due to velocity	--
R	Gas constant (for air: 2.47×10^5)	$in^2/sec^2\text{-}°R$

SYMBOL	DEFINITION	UNITS
r_1, r_2	Complex pair of roots in the s-plane	
r_3, r_4, r_5	Negative real roots in the s-plane (r_3 is the root closest to the origin and r_5 is the root most remote from the origin)	
$\underline{s}$	Laplace operator	
t_c	Capillary passage thickness	in
T_s	Temperature of working fluid	°R
V_t	Volume of each accumulator bottle	in^3
V_i	Cylinder volume when ram is centered	in^3
w_a (w_b)	Weight of air in chamber a (or b)	lb
w_c	Capillary passage width	in
X_a	Linear travel of the point on the door extended to (h + c) inches from hinge line	in
X_b	Feedback connecting rod displacement	in
X_c	Linear travel of the point on the door extended to c inches from the hinge line	in
X_o	Piston rod travel	in
X_s	Damper arm displacement	in
X_v	Valve arm displacement	in
X_i	Input displacement	in
$\Delta(s)$	Characteristic equation	--
μ	Absolute viscosity of air	$lb\text{-}sec/in^2$
τ	Transient pressure feedback time constant	sec
ζ	Damping ratio	--
G_1	$(a + b)h/ae$	--

SYMBOL	DEFINITION	UNITS
G_2	c/d	--
G_3	$(h + c)f/ed$	--

SUMMARY

There is a growing demand today for high performance equipment which can respond precisely to any input in a speed and accuracy range far beyond the capability of the manual control of a human operator. The steering mechanism in the booster of a space craft, while it is still in the earth's atmosphere, needs a concentration of high power, precisely controlled and contained in a small area. Many of these use a compressed gas, not necessarily air, for their steering mechanism.

The purpose of this study is to present an extension of the Root Locus Method, developed by Evans (5), from the single parameter study to an _n_ parameter study, so that the systems behavior may be predicted by observing its root variation in a single synoptic plot in the complex plane as its _n_ parameters are varied. All parameters in the system, which were to be optimized for dynamic characteristics, were preassigned practical limits. This made it possible to form a closed contour plot in the complex plane.

In order to illustrate the method of analysis suggested above, a particular system was postulated. The technique will be presented in terms of a specific example carried through to completion, rather than in general terms. The proposed system selected was the steering mechanism on a rocket booster, which is presented schematically in Figure 1. Let it be assumed that the actuator size, operating pressure and temperature, valve travel, inherent damping, reaction load, mass of load and basic geometry were previously determined or specified. Therefore, the

analysis evolved to the following objectives:

1. To find compatible linkage gain values of the forward and feedback mechanisms, so that with the valve and actuator centered, a hard-over input caused a full-over valve. Then as the actuator, in response to this input, left center to full-over, the valve returned to center while the input remained hard-over.

2. To find the optimum valve characteristic, so that a hard-over to the input caused the actuator to move from centered position to hard-over against a full load in one second.

3. To find the method of governing the response time irrespective of load so that the principal time constant is one second. This implies some type of velocity feedback.

4. To investigate the damping of the oscillatory mode so that the damping ratio (ζ) is equal to or greater than 0.5.

The system was analyzed by the root locus method. Results indicated that the system required a velocity limiting feedback for two reasons: (1) to restrain the response time to one second as specified, and (2) to prevent the system from destroying itself under a no-load configuration.

The uncorrected system was inherently underdamped: therefore, external damping was required. Plenum bottles were added to the system to increase damping.

All analyses were based on in-flight atmospheric conditions, with one exception. An investigation of K_L showed that when the booster was not in flight (i.e., K_L set equal to zero) the principal time constant decreased slightly while the damping ratio increased slightly. Briefly this means that when the booster is actuated in static air, the response time is slightly faster--and the oscillatory mode is more nearly critical--than that experienced in full atmospheric flight.

CHAPTER I

INTRODUCTION

Definition.--A system is a logical collection of components which is arranged in such an order as to cause a predetermined output behavior, given some particular input. Thus the basic equipment plus its control loop makes up the system.

The precision and high speed equipment used in modern technology have largely exceed man's ability to control manually. When accuracy in positioning and precision speed are coupled with large forces, some type of automatic control must be added. In the position system used throughout this study all the factors, precision in positioning, speed, and large forces are present. The equipment needs an automatic control primarily for control in flight, but equally essential is some type of control to prevent self-destruction, due to the large forces, under no-load conditions. The self-destructive characteristic is more evident in this system than, say, in a stamping machine. Both require precision and both use large forces, but in the stamping machine design more structural parts generally may be added in weaker areas, while the steering mechanism must be designed as lightly as possible consistent with good engineering practice.

Historical Note.--The use of pressurized fluids is not new. During the Industrial Revolution in England vast networks of lines were used to supply energy to the textile mills of that time. Those designers developed and used many of the components used today, including pumps, compressors, valves and surge tanks. The advent of electrical power transmission reduced the use and hence the continuing development in pressurized fluid applications.

Lately this science was revived by the demand of designers of aircraft, ocean craft, missiles and other devices. It satisfies the requirements for large forces in a small package. The task of precisely controlling these forces falls on the shoulders of the controls engineer.

Before 1932, systems were generally developed by a trial and error technique based on an extension of previously successful designs. In some of the sophisticated approaches to the earlier problems, the defining equations were developed for each component, just as designers do today, except that they remained always in the time domain and reduced these time-dependent equations to some workable form. This was tedious and led only to limited conclusions even for relatively simple systems.

Nyquist (4) in 1932 published his frequency response method, which uses the Laplace Operational Calculus in conjunction with Cauchy's Theorem found in Function of Complex Variable Theory. Cauchy's theory, briefly stated, says that in mapping the boundaries of a closed contour in one complex plane through some mapping function to a second complex plane, the behavioral pattern of the tracing in the second complex plane indicates the presence or absence of singularities in the closed contour in the first complex plane. Thus, one using the Nyquist approach would map the right half of the s-plane through the systems characteristic equation to the Nyquist plot. Should the Nyquist plot indicate singularities, they then exist in the right half of the original s-plane. This is tantamount to saying that in the time domain one or more of the exponents of the exponential e is positive and, hence, the system is unstable. The actual interpretation of the Nyquist plot will not be discussed here, but reference will be made to Brown and Campbell (4).

The frequency response developed by Nyquist was an important analysis technique. His method is used extensively today in nearly all analysis work, particularly in the field of filter theory in which noise plays an important part. However, the writer is not alone in the belief that in the design and analysis of most electro-mechanical or mechanical systems there exists a superior method of analysis, the Root Locus Method.

Root Locus Method.--The purpose of this work is to illustrate, by means of the postulated system, an extension of the Root Locus Method. Evans (5) developed his method during the Second World War and published it in 1947. The extension to his work undertaken here is the extension of a single variable to n-variables, so that a designer may infer the system's transient behavior by observing the changes of n-parameters in a synoptic contour plot. This tacitly means he must, in some cases, visually interpolate the magnitudes of the n-parameters so that the particular root remains in some preassigned area in the contour plot. The fact that n-parameters may be permitted to range between their respective prescribed limits is a distinct advantage over the single parameter variation developed by Evans. Incidentally, his has a distinct advantage over the Nyquist approach where no parameter is allowed to vary. In the Nyquist technique all the parameters are fixed with a set of assigned values, then laborously tested for stability by imposing a sinusoidal input with frequency, ω, the independent variable. In addition to the advantage discussed above, the Root Locus Method is more direct and permits the designer to see the system's transient behavior at all times. The direct relationship between parameter magnitudes and root location is always apparent in the contour plot. The relationship between root location and transient behavior is read off this same plot in terms of damping and frequency of the system. Thus, for a particular damping and response time constant, the parameters are varied until the roots fall in the required area. The parameter values are then read off and in turn define the sought-for system.

CHAPTER II

METHOD OF ANALYSIS

Problem Restated.--The primary objective of this study is to present a method for optimizing the parameters of a high-pressure pneumatic piston-operated actuator so that, given an input, it will behave in some prescribed manner irrespective of load.

Rather than present the techniques in general terms, a particular system with its own special requirements was postulated. Let it be assumed the system's basic dimensions and load, which are listed below, are initially specified. The analysis then developed into the following tasks:

a) Find the linkage ratios in the forward and feedback loops.

b) Optimize the proportional control valve characteristics.

c) Devise and optimize an acceptable external velocity limiting mechanism, so that the actuation time from centered to full-over will be one second, irrespective of load.

d) Devise and optimize an acceptable external damping device, if required, to nullify high frequency chatter. Let it be specified that the damping ratio be equal to or greater than 0.5.

Basic Approach to the Problem.--The linkage ratios were determined by kinematic considerations only. The valve characteristics were defined by the air flow requirements in Figure 4, which, in turn, were dependent on the specified rate of travel of the actuator. The system's valve travel and output travel were proportional.

The need for a velocity limiting device is particularly important for a high pressure system with its high forces. Without such a device the system, designed to operate under full load and given a hard-over input under a no-load configuration, would slam hard-over and possibly destroy itself. The corrective device used in this study consists of a spring and viscous damper connected into the feedback linkage of the valve.

In the event the system's oscillatory mode is more lightly damped than the specified ratio, $\zeta = 0.5$, and cannot be corrected by the valve characteristic, k_2, then plenum or accumulator bottles will be connected to either chamber by capillary flow passages.

Before any of the above could be accomplished, the defining equations, transfer function, and the characteristic equation were first formulated. The non-bottle configuration was examined first.

Postulated Dimensions of the System.--The values listed below are the product of the writer's experience on previous systems. While they are not from any particular system, they were used to illustrate the method of analysis.

P = 3750 lbs/in^2, system's supply pressure

A = 10 in^2, area of the piston head

m_L = 0.1 lb-sec^2/in, mass equivalent of the airfoil fluid, and connections

C_L = 20 lb-sec/in, system's total damping, air viscous damping and damping due to bearings

K_L = 5,000 lbs/in, spring rate of air load. Assumed constant for the amount of travel.

<u>System Parameters to be Optimized</u>.--The system's remaining parameters reduced to:

a,b,c,d,e,f,g, and h, linkage values	in
k_1, valve characteristic coefficient denoting no-load flow sensitivity	in^2/sec
k_2, valve characteristic coefficient, denoting loss of flow per unit load pressure	$in^5/lb\text{-}sec$
k_2', transient-pressure-feedback coefficient	$in^5/lb\text{-}sec$
τ, transient-pressure-feedback time constant	sec
k_3, coefficient denoting fluid compliance	in^5/lb
K_s, spring rate of velocity limiter	lb/in
C_s, damping rate of velocity limiter	lb-sec/in
K_s/C_s, ratio of above	sec^{-1}

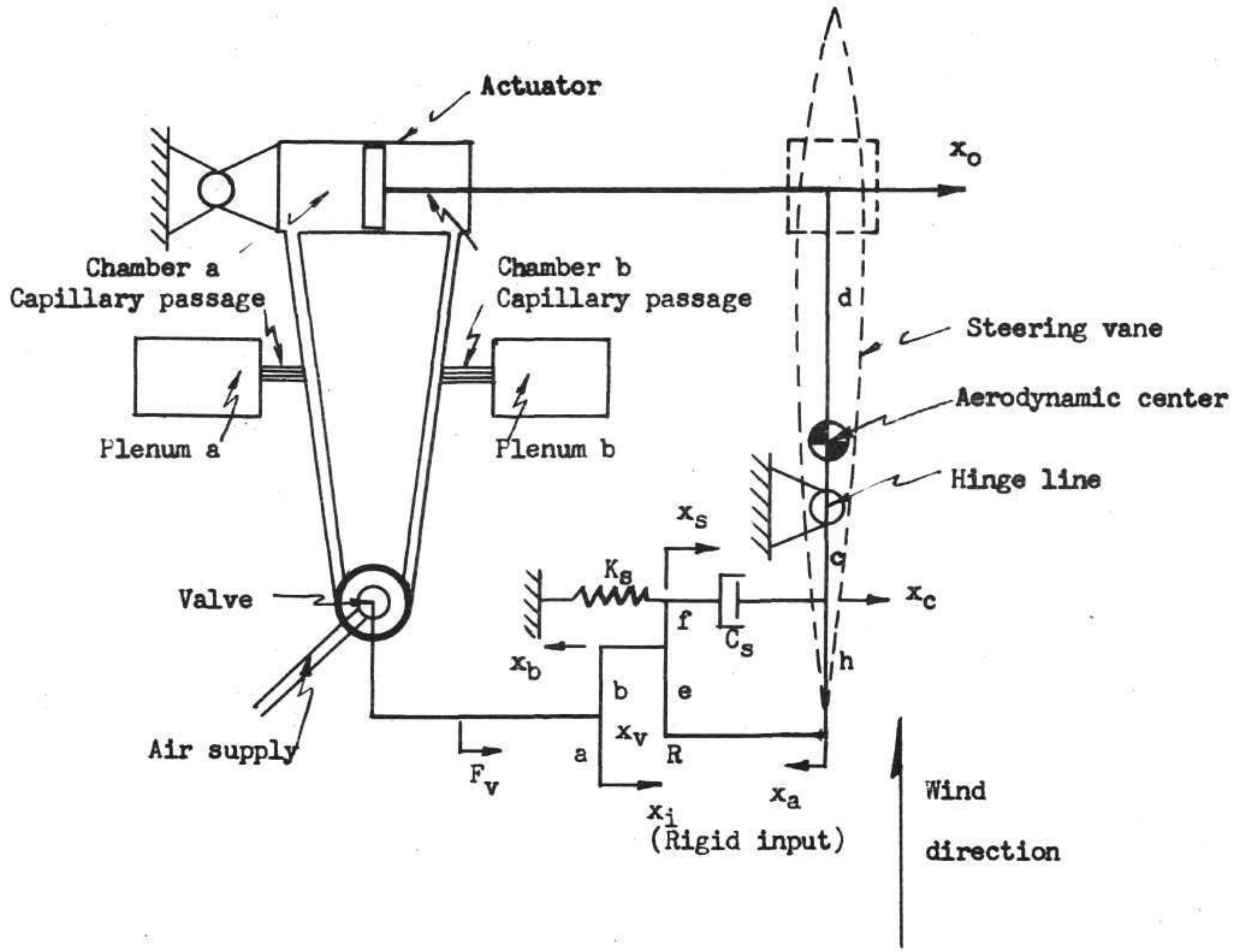


Figure 1. System Schematic Diagram

CHAPTER III

ANALYSIS

Derivation of Fundamental Equations.--Displacement and force equations were formulated from the diagram shown in Figure 1. These, alone with the flow equation developed by Shearer (2), completely define the system analytically.

The two displacement equations, each on either side of the velocity limiter, were summed respectively about the input and air foil hinge line. The movements are not simply connected but are interdependent on the remaining displacements; therefore, their total displacements were found by adding their partials in the following manner:

$$x_i = f(x_v, x_b) \tag{1}$$

$$x_a = g(x_s, x_b) \tag{2}$$

Total derivatives were taken of the equations above.

$$dx_i = (\partial x_i/\partial x_v)dx_v + (\partial x_i/\partial x_b)dx_b \tag{3}$$

$$dx_a = (\partial x_a/\partial x_s)dx_s + (\partial x_a/\partial x_b)dx_b \tag{4}$$

The displacement ratios for each of the four partials were formulated directly from Figure 1.

$$\frac{\partial x_i}{\partial x_v} = \left.\frac{\Delta x_i}{\Delta x_v}\right|_{x_b = \text{constant}} = \frac{a+b}{b} \tag{5}$$

$$\frac{\partial x_i}{\partial x_b} = \left.\frac{\Delta x_i}{\Delta x_b}\right|_{x_v = \text{constant}} = \frac{a}{b} \tag{6}$$

$$\frac{\partial x_a}{\partial x_s} = \left.\frac{\Delta x_a}{\Delta x_s}\right|_{x_b = \text{constant}} = \frac{e}{f} \tag{7}$$

$$\frac{\partial x_a}{\partial x_b} = \left.\frac{\Delta x_a}{\Delta x_b}\right|_{x_a = \text{constant}} = \frac{f+e}{f} \tag{8}$$

The above ratios were substituted into equations (3) and (4). If displacements are conservative, the linkage movements may then be assumed linear. That is, equations (3) and (4) may be written as follows:

$$x_i = \frac{(a + b)}{b} x_v + \frac{a}{b} \quad x_b \tag{9}$$

$$x_a = \frac{e}{f} x_s + \frac{(f + e)}{f} \quad x_b \tag{10}$$

Again from Figure 1,

$$x_a = \frac{(c + h)}{d} \quad x_o \tag{11}$$

Equation (11) was substituted into equation (10) and rearranged to the following,

$$x_o = \frac{d}{h + c} \frac{e}{f} x_s + \frac{h}{f} \frac{d}{h + c} x_b \tag{12}$$

As a result of taking moments about the point of input, the force equation evolved to the following:

$$(a + b)\ F_b = (a)\ F_v. \tag{13}$$

And as a result of taking moments about the point R (see Figure 1), a second force equation evolved to

$$(e)\ F_b = -(h)\ F_s \tag{14}$$

Summing forces and displacements about the direction of x_s,

$$F_s = K_s X_s + (\dot{X}_s - \dot{X}_c) C_s \tag{15}$$

$$X_c/c = -X_o/d \tag{16}$$

from which,

$$\dot{X}_c = -\dot{X}_o c/d \tag{17}$$

Equation (17) was substituted into (15),

$$F_s = X_s K_s + C_s \dot{X}_s + \dot{X}_o C_s c/d \tag{18}$$

Summing air flow (1),

$$A\dot{X}_o = k_1 X_v - k_2 P_m - k_3 \dot{P}_m \tag{19}$$

Forces were summed at the actuator,

$$P_m A = m_L \ddot{X}_o + C_L \dot{X}_o + K_L X_o \tag{20}$$

Equations (9), (12), (13), (14), (18), (19), and (20), which define the system, were rewritten and placed in Laplace operational form.

---

(1) Blackburn, Reethof, Shearer, Fluid Power Control, p. 546.

The following were used to convert the time dependent equations to the s-domain.

The Laplace of X(t) was written in the usual form:

$$\mathcal{L}\, X(t) \;=\; x(s),$$

$$\mathcal{L}\, \dot{X}(t) \;=\; sx(s) - X(0)$$

$$\mathcal{L}\, \ddot{X}(t) \;=\; s^2x(s) - sX(0) - X(0)$$

It should be stated at this point, that since only the characteristic equation is required, the initial conditions were set equal to zero, $(X(0) = \dot{X}(0) = \ddot{X}(0) = 0)$. Also for simplicity, x(s) was written as x alone; the argument s always being implied,

$$x_i \;=\; x_v(a + b)/b + x_b a/b \tag{21}$$

$$x_s \;=\; -x_b h/e + x_o f(h + e)/ed \tag{22}$$

$$F_b(a + b)/a \;=\; F_v \tag{23}$$

$$F_b \;=\; -F_s h/e \tag{24}$$

$$F_s \;=\; (K_s + sC_s) + sC_s x_o e/d \tag{25}$$

$$P_m A^2 = (m_L s^2 + C_L s + K_L)x_o \tag{26}$$

$$Ax_o s \;=\; k_1 x_v - (k_2 + k_3 s)P_m \tag{27}$$

The overall transfer function was found by setting fundamental equations (21) through (27) in a square matrix array and solving for $x_o(s)$ as a function of $x_i(s)$. The transfer function is the ratio of output over input, $x_o(s)/x_i(s)$, in Laplace operational form.

$$\begin{bmatrix} 1 & (a+b)/a & 0 & 0 & 0 & 0 & 0 \\ -h/e & 0 & 0 & 0 & -1 & (h+e)f/ed & 0 \\ 0 & 0 & 0 & -(a+b)/a & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & h/e \\ 0 & 0 & 0 & 0 & (K_s+sC_s) & sC_s c/d & -1 \\ 0 & 0 & -A & 0 & 0 & (m_L s^2+C_L s+K_L) & 0 \\ 0 & k_1 & -(k_2+k_3 s) & 0 & 0 & -As & 0 \end{bmatrix} \begin{bmatrix} x_b \\ x_v \\ P_m \\ F_b \\ x_s \\ x_o \\ F_s \end{bmatrix} = \begin{bmatrix} (b/a)x_i \\ 0 \\ -F_v \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The solution for x_o yielded the following equation,

$$x_o = \frac{Ak_1(K_s + sC_s)b/(a + b)x_i - Ak_1(d/c(1/G_1^2)F_v}{\Delta(x)} \qquad (28)$$

where

$$\Delta(s) = Ak_1(K_s + sC_s)G_3/G_1 + Ak_1C_s sG_2/G_1 + \left[A^2 s + (m_L s^2 + C_L s + K_L)(k_2 + sk_3)\right](K_s + sC_s) \qquad (29)$$

and

$$G_1 = (a + b)h/ae, \quad G_2 = c/d, \quad G_3 = f(h + c)/ed$$

Transfer Function.--"The transfer function of a filter is the ratios of the Laplace transform of any normal response and the input that produced it" (8). From a mathematical viewpoint either x_o/s_i or x_o/F_v is the transfer function. In either case the denominator is the characteristic equation, which is the primary equation in this parameter study. Equation (28) may be used to determine the time history of the output after all parameters are finalized. This was done at the end of the study to illustrate the relationship between the system's parameters, interpolated from the contour plots in the complex plane, and the coefficients and exponents of the time history curve, for a given input, $x_i(s)$.

Also equation (28) may be used with the Final Value Theorem (4) as stated by equation (30) to check the accuracy of the linkage values and other computed systems parameters.

$$\lim_{t \to \infty} X(t) = \lim_{s \to 0} sx(s) \tag{30}$$

Let the input be a step input,

$$x_i(s) = |x_i|/s \tag{31}$$

The coulomb force, F_v, has a degrading effect on the system behavior, but this can be reduced to an acceptable level by increasing the value of K_s, the spring constant of the velocity feedback damper. This, in turn means C_s must be increased proportionally since the ratio K_s/C_s is one of the parameters to be determined. Therefore, if F_v is assumed small, then,

$$x_o = \frac{Ak_1 b|x_i|G_1}{(a + b)(Ak_1 G_3 + K_L k_2 G_1)} \tag{32}$$

The actual value of K_s will be determined in a later section.

Characteristic Equation.--The solution of the square array in the above determinant equated to zero is the characteristic equation. It is also the denominator of equation (28). Normalizing and simplifying, the coefficients of the characteristic equation, in descending order of the operator s, reduced to,

$$
\begin{array}{ll}
s^4 & 1 \\[1em]
s^3 & (k_2/k_3 + C_L/m_L + K_s/C_s) \\[1em]
s^2 & \left[(C_L/m_L)(k_2/k_3) + K_L/m_L + A^2/m_L k_3\right] + (k_2/k_3 + C_L/m_L)K_s/C_s \\[1em]
s^1 & (K_L/m_L)(k_2/k_3) + \left[(C_L/m_L)(k_2/k_3) + K_L/m_L + A^2/m_L k_3\right]K_s/C_s \\
 & \qquad + (Ak_1/m_L k_3)(G_2 + G_3)/G_1 \\[1em]
s^0 & (K_L/m_L)(k_2/k_3)(K_s/C_s) + (Ak_1/m_L k_3)(G_3/G_1)(K_s/C_s)
\end{array}
$$

<u>Determination of Linkage Values</u>.--There is no unique set of linkage lengths on the forward and feedback linkages, but a compatible set of values was calculated as follows,

Equation (11) was substituted into (22) and the resultant substituted into (21),

$$x_s = -(hb/ea)x_i + \left[h(a + b)/ea\right] x_v \tau (f/e)x_a \qquad (33)$$

The boundary conditions were assumed as follows,

$$|x_o|_{max} = 5 \text{ inches} \qquad (34)$$

Hinge line travel = ±18° and input to valve travel = ±18°

$$|x_a|_{max} = (h + c) \sin 18° = (h + c)(0.309) \qquad (35)$$

$$|x_i|_{max} = (a+b)(g/b)\sin 18° = (a+b)(g/b)(0.309) \qquad (36)$$

Substitute (35) and (36) into (33), with $x_v = x_s = 0$

$$0 = \left[(-hb)/(ae)\right]\left[(a + b)g/b\right](0.309) + \left[(h + c)(f/e)\right](0.309) \quad (37)$$

Simplifying (37)

$$(a + b)hg/a = f(h + c), \text{ provided } e \neq 0 \quad (38)$$

If a = b, then (arbitrarily chosen since there is no unique solution),

$$2hg = f(h + c) \quad (39)$$

$$|x_o|/d = \sin 18° \quad (40)$$

$$c = 5/(0.309) = 16.16 \text{ inches} \quad (41)$$

If f = 2, h = 6 and c = 6, (arbitrarily chosen) then from (39)

$$g = 2$$

Recapping the values thus far determined,

$$a = b = 3$$

$$c = h = 6$$

$$d = 16.16$$

$$e = 4$$

$$f = g = 2$$

From (35) and (36)

$$|x_a|_{max} = 12(0.309) = 3.71 \text{ inches} \tag{42}$$

$$|x_i|_{max} = 4(0.309) = 1.236 \text{ inches} \tag{43}$$

One check on the compatibility of the linkage values is in the use of the Final Value Theorem indicated in equation (31). In a static check or a long time interval after a given input, the position of the vane is affected by the external load, K_L. With $K_L = 0$ the vane should travel full over, given a full over input. Therefore, with K_L set to zero in equation (32)

$$|x_o|_{max} = \frac{b|x_i|G_1}{(a + b)G_3} = \frac{(3)(1.236)(3)}{(6)(0.371)} = 5 \text{ inches} \tag{44}$$

which checks the value indicated in equation (34). The Final Value Theorem serves as a necessary but not a sufficient check on the compatibility of all linkage values, both assumed and computed. The final and sufficient check on compatibility of all the above values and all those yet to be determined were checked below in the section on Time History of the System.

<u>Determination of K_s/C_s</u>.--The values of K_s and C_s were evaluated in two steps, statically in which K_s alone was computed and dynamically in which C_s was computed using the previously determined value of K_s.

K_s was determined as follows. From equations (23), (24), and (25) and setting $sC_s = 0$,

$$\left[-h(a + b)/ea\right] K_s x_s = -F_v \tag{45}$$

$$0.333F_v = K_s x_s \tag{46}$$

Let it be required that a five per cent displacement on x_i, which is transmitted to the velocity feedback damper by the linkages, will be just sufficient to overcome the reaction force due to valve coulomb friction.

Substitute equation (16) into equation (33)

$$x_s = -1.5\, x_i + 3.00\, x_v + 0.375\, x_o \tag{47}$$

Letting $x_v = x_o = 0$

$$x_s = -1.5\, x_i \tag{48}$$

From the calculations of the displacement required to cause the valve coulomb friction force to be equal to the reaction force due to $K_s x_s$, and with,

$$x_i = 2|x_i|\ 5\% = 0.12 \text{ inches} \tag{49}$$

Then

$$x_s = -0.18 \text{ inches} \tag{50}$$

Substitute equation (50) into (46)

$$K_s = -(0.333)F_v/(0.18) \tag{51}$$

If F_v (the coulomb force required to break away the valve) = 12.5 lbs, then,

$$K_s = (0.33)(12.5)/(0.18) = 23.1 \text{ lbs/in} \tag{52}$$

The following determination of C_s is based on empirical and assumed data and maximum velocity of actuator, $|\dot{x}_o|_{max} = 10$ in/sec. The lowest acceptable value is, $|\dot{x}_o|_{max} = 5$ in/sec. These values were based on the postulated full travel from stop to stop of one to two seconds. The value of C_s was computed using the assumed conditions as follows. It was assumed that at the instant of investigation the vane was at midpoint and was travelling at the maximum (and also at the minimum) velocity and that the force due to C_s and K_s was equal to the reaction force due to coulomb friction. From equations (23), (24), and (25) in the time domain,

$$(0.333)F_v = K_s x_s + C_s \dot{x}_s + C_s \dot{x}_c \tag{53}$$

The assumptions indicated above were written in equation form below,

$$F_v = 12.5 \text{ lbs (assumed)}$$

$$\left.\begin{aligned} x_s &= R_1 \dot{x}_c, \text{ where} \\ R_1 &= -0.20 \text{ sec} \end{aligned}\right\} \text{Empirical}$$

$$1.88 \leq x_c \leq 3.72 \text{ in/sec} \quad \text{(limits on original postulated system)}$$

$$\dot{x}_s = \ddot{x}_c = 0 \text{ at mid-stroke}$$

Substituting the above into equation (53) yields, after simplifying,

$$|C_s|_{max} = 6.87 \text{ lb-sec/in}$$

and similarly,

$$|C_s|_{min} = 5.75 \text{ lb-sec/in}$$

Therefore, the required ratios are,

$$K_s/C_s = 23.1/6.87 = 3.36 \text{ sec}^{-1} \tag{54}$$

and,

$$K_s/C_s = 23.1/5.75 = 4.02 \text{ sec}^{-1} \tag{55}$$

The values of K_s/C_s above are intended to determine the approximate area of investigation. The finalized value was taken from the contour plot in Figure 10.

Valve Characteristics, k_1 and k_2.--Let the valve have the following properties:

(a) Four way. One pair of variable area orifices, supply and exhaust, was connected to either side of the cylinder chamber.

(b) No valve underlap or overlap. Ideally this means when the valve is centered, there is no leakage. Actually, there is leakage and most valve designers assume the leakage is such that $P_a = P_b = 0.667P_s$, when the ram is centered and under no load.

(c) The ratio of actuator travels to valve travel is constant: $|x_o/x_v|$ is constant under the above conditions.

To find the port areas, the weight rate of air must be computed so that with the ram centered and under quiescent conditions, a full over signal given to the valve drives the ram to full over in one second against the load, $K_L x_o$. The amount of air in either chamber at the end of travel, minus the air initially, is the amount of air added. Using the ideal gas law,

$$PV = w_a RT/g, \text{ yields} \tag{56}$$

$$w_a = g\left[2500(110) - 14.7(60)\right]/RT \tag{57}$$

$$w_a = 0.764 \text{ lbs}$$

Even though the air flow rate is not constant throughout the interval of one second, it is fairly constant for the greater part of the flow time history curve, due to critical or choked flow. Therefore, let it be assumed the rate is constant throughout, and that,

$$\dot{w}_a = 0.764 \text{ lbs/sec} \tag{58}$$

The postulated valve is linear; that is, the weight rate of flow into chamber a (or b) is directly proportional to valve position, and to the valve rectangular port area, for steady-state and no load conditions. k_1 is a function of weight rate of flow.

The weight rate of air flow through the valve's rectangular port a (or b) is proportional to its area, A_o, the flow factor, $f(P_a/P_s)$, and the discharge coefficient, C_D. For a given P_a/P_s there exists a $f(P_a/P_s)$ and a C_D which were read directly from Figure 2 and Figure 3, respectively, and substituted into the weight rate of flow given below

$$\dot{w}/A_o = 0.528(C_D)(P_s)\ f(P_a/P_s)/\sqrt{T} \tag{59}$$

where

$$P_s = 3750 \text{ lbs/in}^2$$

$$T = 560° \text{ R}$$

$$A_o = \text{Orifice area, a function of } x_v$$

C_D = Discharge coefficient of air, Fig. 2

$f(P_a/P_s)$ = Pressure coefficient, Fig. 3.

The calculated values for $\dot{w}_a(t)$, valve position and upstream pressure were plotted in Fig. 4.

The values of k_1 and k_2 were computed from the following equations, taken from Reference 2, page 245,

$$k_1 = (\partial \dot{w}_a/\partial x_v)\ RT_s/gP_i \tag{60}$$

$$k_2 = (\partial \dot{w}_a/\partial P_m)\ RT_s/gP_i \tag{61}$$

where,

$$R = 2.47 \times 10^5$$

$$T_s = 560\ ^\circ R$$

$$g = 386 \text{ lb-sec}^2/\text{in}$$

$$P_i = 0.667\ (P_s) = 2500 \text{ lbs/in}^2$$

From Fig. 4,

$$\partial \dot{w}_a/\partial x_v\Big|_{max} = 0.764/0.618 = 1.236 \tag{62}$$

and

$$\partial \dot{w}_a/\partial P_m = 1.65 \tag{63}$$

The constant terms in equations (60) and (61) reduced to,

$$RT_s/gP_i = 143.3,$$

Therefore, the required valve constants were,

$$k_1 = 177.2 \text{ in}^2/\text{sec} \tag{64}$$

$$k_2 = 0.0236 \text{ in}^5/\text{lb-sec}$$

Determination of k_3.--The coefficient denoting fluid compliance is a direct function of the volume of the compressed column. Let it be assumed the fluid flow lines are short from the valve to chambers a (or b) (4).

$$k_3 = V_i/2kP_i \tag{65}$$

where,

$$V_i = 60 \text{ in}^3$$

$$k = 1.4\text{, for air}$$

$$P_i = 2500 \text{ lbs/in}^2$$

Substituting known values into equation (65) results in,

$$k_3 = 8.6 \times 10^{-3}\text{, actuator centered} \tag{66}$$

Eigenroots of the Characteristic Equation.--The analysis was simplified by concentrating on,

a) the oscillatory mode first and investigating any corrective action that the system might require; then

b) concentrating on the response time characteristics and

(4) Shearer, J. L., "Study of Pneumatic Processes in Continuous Control of Motion of Compressed Air," p. 245.

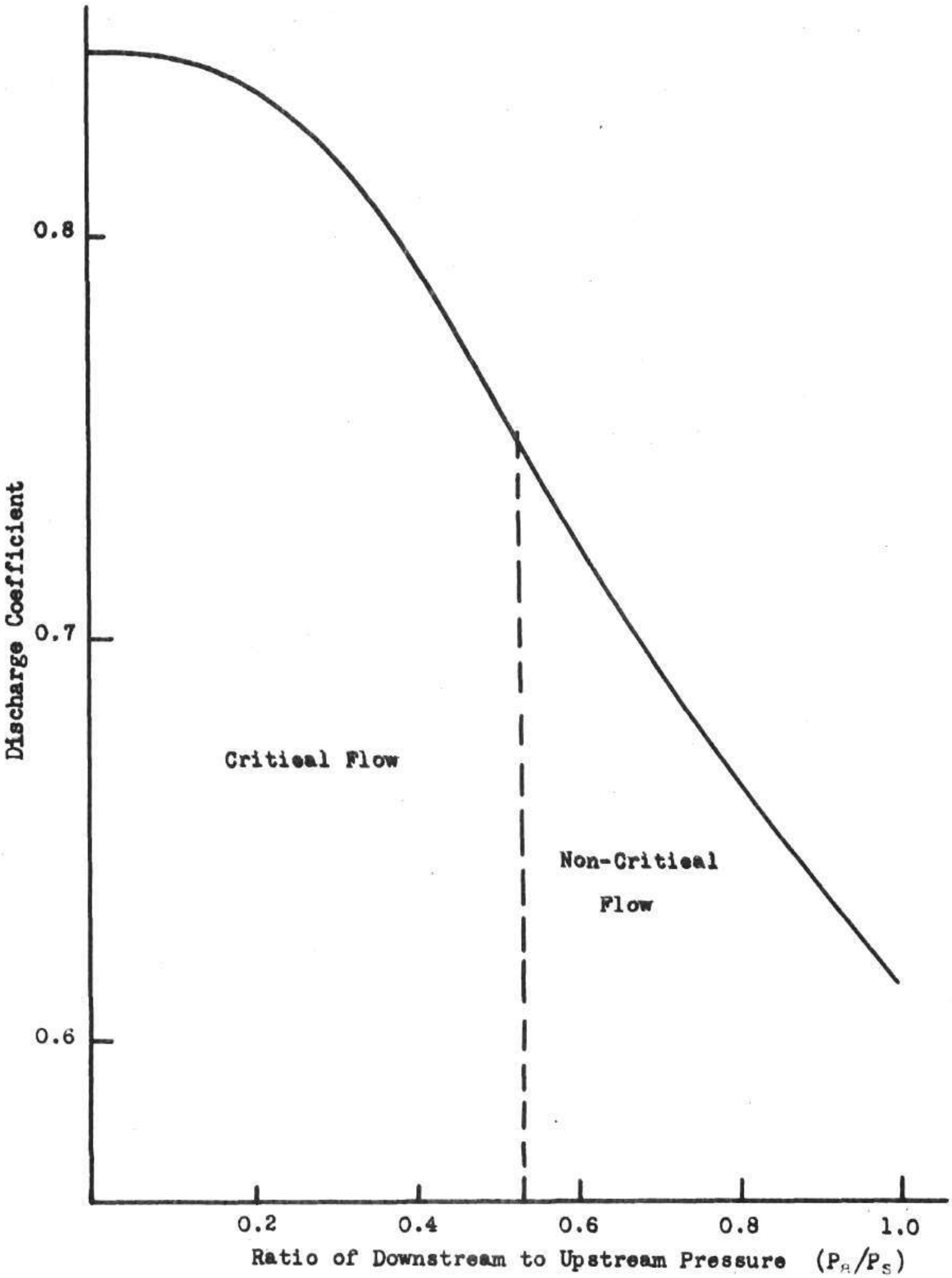


Figure 2. Discharge Coefficient for Air

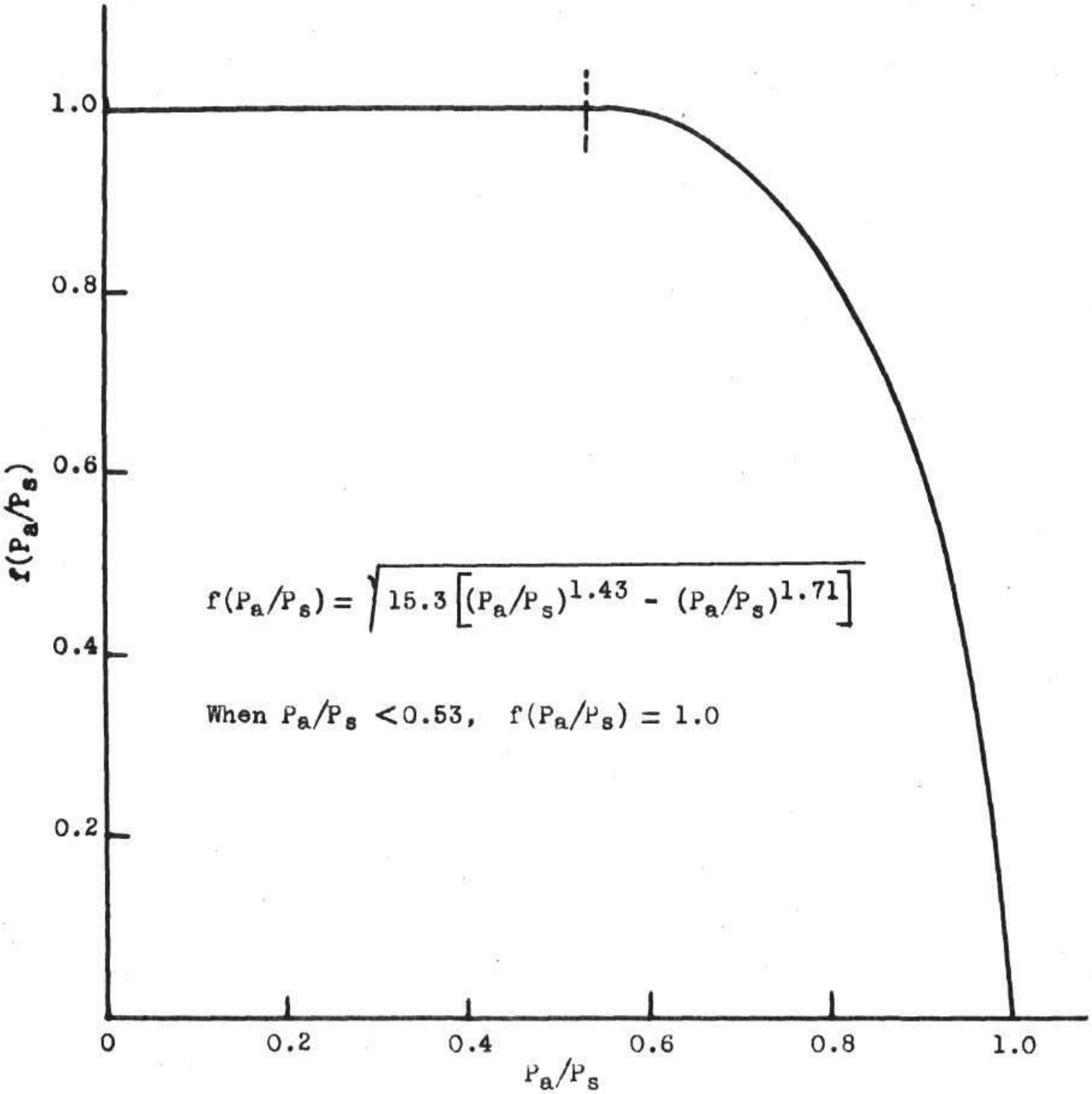


Figure 3. Non Dimensional Flow vs. Downstream to Upstream Pressure Ratio

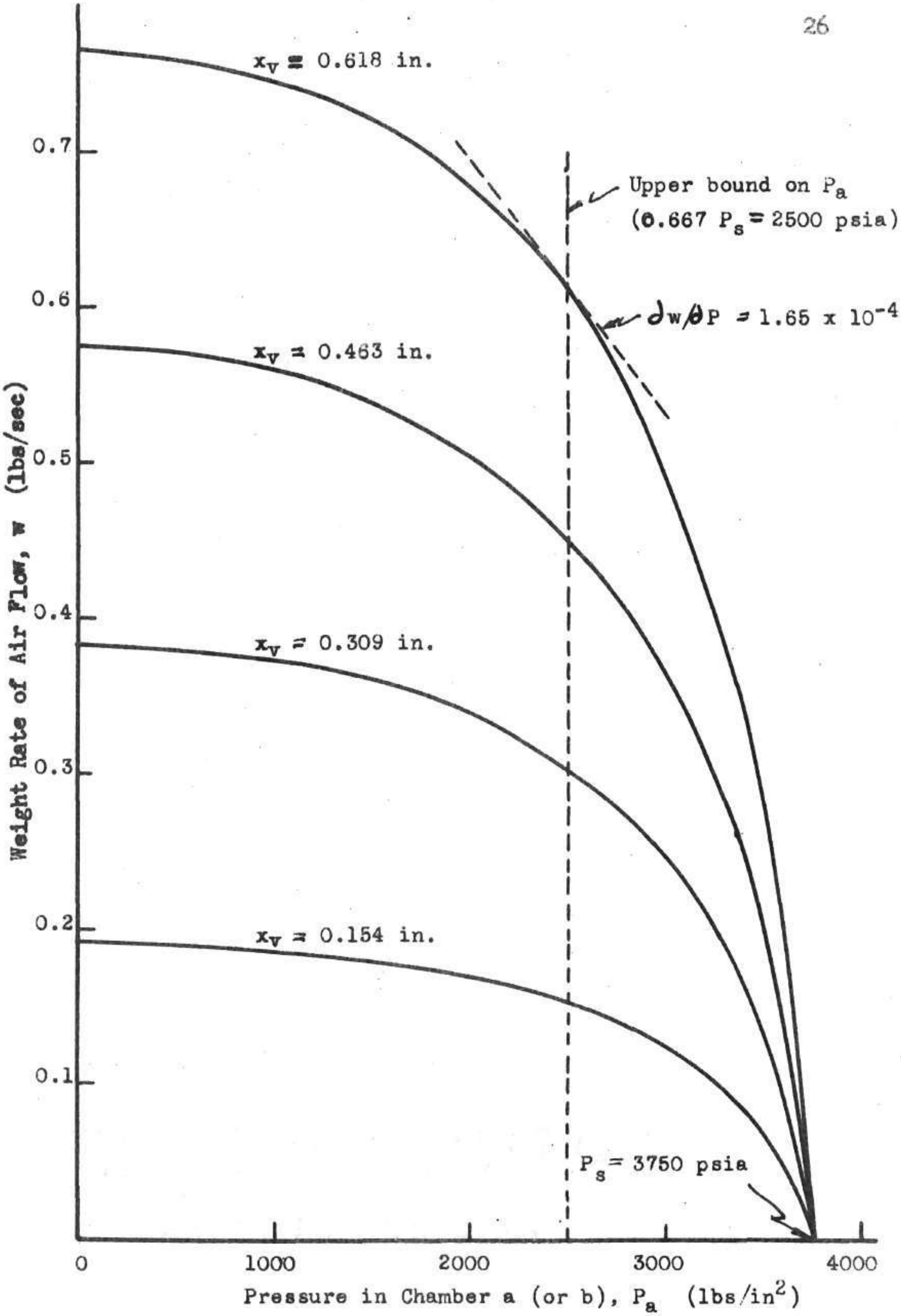


Figure 4. Valve Characteristics of Postulated Valve

investigating any corrective action they might require.

In mathematical terms, this means looking first at the complex pair of roots graphically in the complex plane to see if they fall to the left of the $\zeta = 0.5$ boundary. If the system is insufficiently damped, that is, falls to the right of $\zeta = 0.5$ boundary, then the amount of damping required to shift the roots over into the acceptable region must be determined. The negative real root nearest the origin in the complex plane, was studied and the parameter, or parameters, which most affected this root were adjusted so that this particular root fell on $r_3 = 1.0$. It is shown in Figure 12 that the smaller negative real roots contributed practically nothing to the time response requirement. Therefore, when $r_3 = 1.0$, the system has an overall time constant of $1/r_3 = 1$ second, the postulated requirement.

In the first investigation K_s/C_s and k_1 affected only slightly the oscillatory mode. Hence, in this phase of investigation these two parameters were set equal to zero. This reduced the characteristic equation from fourth to third order.

All known values were substituted into the coefficients of s found in the section, "Characteristic Equation," pages 14 and 15. The knowns were rewritten for convenience,

$$K_s/C_s = k_1 = 0$$

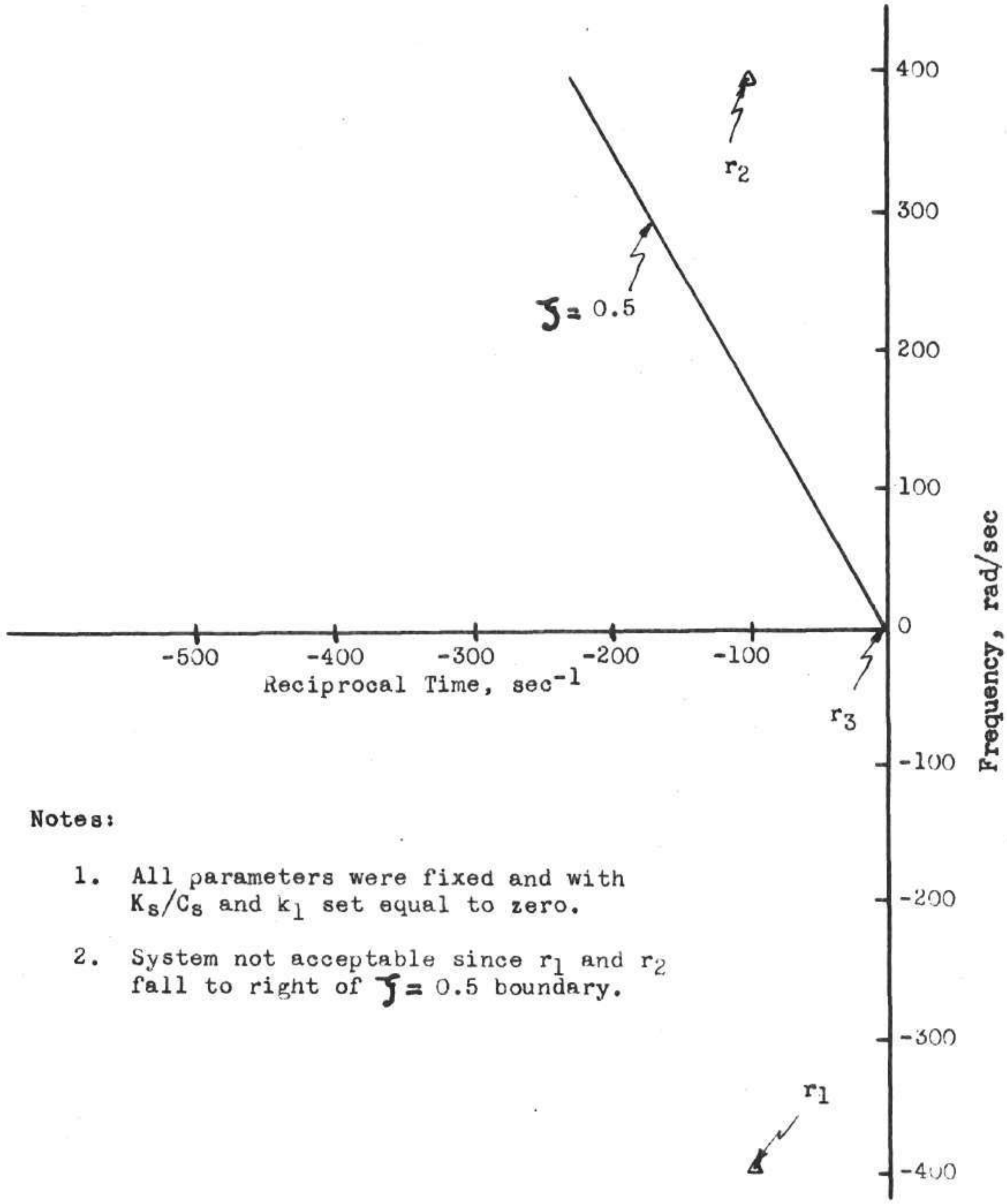
$$k_2 = 0.0236 \text{ in}^5/\text{lb-sec}$$

$$k_3 = 8.6 \times 10^{-3}$$

$$C_L = 20 \text{ lbs-sec/in}$$

$\Delta(s) = (s + 0.918)(s + 101.1 \pm 395.5j)$

Test 1



Notes:

1. All parameters were fixed and with K_s/C_s and k_1 set equal to zero.

2. System not acceptable since r_1 and r_2 fall to right of $\zeta = 0.5$ boundary.

Figure 5. Basic System Oscillatory Mode Investigation

$$K_L = 5000 \text{ lbs/in}$$

$$A = 10 \text{ in}^2$$

$$M_L = 0.1 \text{ lbs-sec}^2\text{/in}$$

$$G_1 = 3.0$$

$$G_2 = G_3 = 0.371.$$

The sought-for characteristic equation reduced to,

$$\Delta(s) = (s = 0.918)(a + 101.07 \pm 395.5j). \tag{67}$$

which, when plotted in Figure 5, indicates that the complex pair falls to the right of the $\zeta = 0.5$ boundary. The plot is different from those which follow in that there is no varying parameter and hence, only points are indicated. The plot shows that the oscillatory mode of the system is inherently underdamped, and can be corrected by external means only.

Dynamic Analysis of System with Plenum Bottles.-- An increase in damping may be effected by the use of accumulator or plenum bottles of suitable size connected each to either side of the valve or piston head through a capillary passage. The addition of the bottles and flow resistances introduces a stabilizing transient flow to each of the tanks when the ram pressures are changing rapidly. The tank configuration changed the characteristic equation from fourth to fifth order, which means a new root was added.

From equation (38) in Reference 2,

$$k_1 x_v - k_2 P_m - k_2' \tau s P_m/(\tau s + 1) - Asx_o = k_3 s P_m \tag{68}$$

The above equation was written directly in Laplace operational form. The new constants were defined as follows,

k_2', equivalent k_2, transient-pressure feedback coefficient.

$$k_2' = C_c/2 \tag{69}$$

τ, Transient-pressure feedback time constant, seconds.

$C_c = w_c t_c^3/12\mu L_c$, capillary resistance coefficient, $in^5/lb\text{-}sec$

μ, absolute viscosity of air 2.62×10^{-9} $lb\text{-}sec/in^2$

w_c, capillary passage width in

t_c, capillary passage thickness in

L_c, capillary passage length in

$$\tau = V_t/kC_cP_i \tag{70}$$

where, V_t = volume of each accumulator bottle

k = ratio of specific heats of air, c_p/c_v

P_i = initial pressure.

Equation (68) was rearranged to the following,

$$Ax_o s = k_1 x_v - \left[k_2 + k_2' s/(s + 1/\tau) + k_3 s\right] P_m \tag{71}$$

The above equation is exactly like equation (27) except that the term $k_2's/(s + 1/\tau)$, was substituted for the k_2. The other equations, (21) through (26), remain unchanged, and when added to equation (71) define the augmented system.

Characteristic Equation for the Augmented System.--The coefficients of the (Laplace) s were recomputed from those of the section "Characteristic Equation," page 14, by using the above substitutions

s^5 $\quad 1$

s^4 $\quad (k_2'/k_3 + C_L/m_L + K_s/C_s) + 1/\tau$

s^3 $\quad \left[(C_L/m_L)(k_2'/k_3) + K_L/m_L + A^2\ m_Lk_3\right] + (k_2'/k_3 + C_L/m_L)K_s/C_s$

$\qquad + (C_L/m_L + K_s/C_s)1/\tau$

s^2 $\quad (K_L/m_L)(k_2'/k_3) + \left[(C_L/m_L)(k_2'/k_3) + K_L/m_L + A^2/m_Lk_3\right]K_s/C_s$

$\qquad + (Ak_1'/m_Lk_3)(G_2 + G_3)/G_1 + \left[K_L/m_L + A^2/m_Lk_3 + (C_L/m_L)\right.$

$\qquad \left.(K_s/C_s)\right] 1/\tau$

s^1 $\quad \left[(K_L/m_L)(k_2'/k_3) + (Ak_1'/m_Lk_3)(G_3/G_1)\right] K_s/C_s$

$\qquad + \left[(K_L/m_L + A^2/m_Lk_3)K_s/C_s + (Ak_1'/m_Lk_3)(G_2 + G_3)/G_1\right]1/\tau$

s^0 $\quad \left[(Ak_1/m_Lk_3)(G_3/G_1)(K_s/C_s)\right]1/\tau$

The numerator of the transfer function reduced to,

$$(K_s/C_s + s)(1/\tau + s)Ak_1b/m_Lk_3\ (a + b)$$

The above are the coefficients of s in the characteristic equation for the augmented system. The addition of bottles increased the weight rate of flow requirements which in turn, meant the original k_1 was now too low. That is, the total volume, chamber a (or b) added to the volume of plenum a (or b), must be pressurized in the prescribed one-second period. The analysis of the augmented system introduced a new negative real root which increased the order of the characteristic equation from fourth to fifth. Let the new k_1 be called k_1'.

Oscillatory Mode Investigation with k_2' and $1/\tau$ Variable.--Preliminary investigation indicated K_s/C_s and k_1' have practically no effects on the oscillatory mode behavior of the system. Therefore, for the oscillatory mode only, K_s/C_s and k_1' were set equal to zero and the coefficients of s in the previous section were simplified to the following.

$$
\begin{array}{ll}
s^3 & 1 \\
s^2 & k_2'(116.3) + 200 + 1/\tau \\
s^1 & k_2'(23{,}260) + 166{,}300 + (200)\ 1/\tau \\
s^0 & k_2'(5{,}815{,}000) + (166{,}300)\ 1/\tau
\end{array}
$$

Accumulator bottles and the connecting capillary flow passages can be designed with k_2' and $1/\tau$ taking on values within a large range. However, any combination of these parameters which causes the complex roots of the system to fall to the right of the $\zeta = 0.5$ boundary is unacceptable according to the original postulated specifications.

Four tests were set up so that k_2' and $1/\tau$, each in its turn, ranged within the limits while the other parameter assumed first its minimum value and then its maximum value. These tests, along with their respective GH(s) functions, are tabulated below.

The results, using the trial sets of values, were plotted in Figure 6 through 9, for the oscillatory mode only. Figures 6 through 9 were then combined into a single plot in Figure 10, which formed a closed contour plot in the complex plane. Only that area within the

Table 1

Test	k_2'	$1/\tau$	GH(s)
2	1	$100 \leq 1/\tau \leq 300$	$\frac{(1/\tau)(s + 100 \pm 395.5j)}{(s + 33.2)(s + 141.6 \pm 400.0j)}$
3	4	$100 \leq 1/\tau \leq 300$	$\frac{(1/\tau(s + 100 \pm 395.5j)}{(s + 120)(s + 272.6 \pm 345.5j)}$
4	$1 \leq k_2' \leq 4$	100	$\frac{116.3k_2'(s + 100 \pm 200j)}{(s + 100)(s + 100 \pm 395.5j)}$
5	$1 \leq k_2' \leq 4$	300	$\frac{116.3k_2'(s + 100 \pm 200j)}{(s + 300)(s + 100 \pm 395.5j)}$

contour and to the left of $\zeta = 0.5$ boundary is acceptable.

Three paired values were chosen along the $\zeta = 0.5$ boundary in Figure 10 for further analysis.

Point A, for which $k_2' = 2.50$ and $1/\tau = 100$

Point B, for which $k_2' = 2.62$ and $1/\tau = 153$

Point C, for which $k_2' = 4.00$ and $1/\tau = 254$

Accumulator Bottle Size.--Those paired values which resulted in the smallest plenum bottles, were selected. Equations 65 and 66 were combined to obtain,

$$V_t = \tau k_2'(2kP_1) \qquad (72)$$

where

$$k = 1.4 \text{ for air}$$

$$GH(s) = \frac{1/\tau\,(s + 100 \pm 395.5j)}{(s + 33.2)(s + 141.6 \pm 400j)}$$

$k_{\frac{1}{2}} = 2$

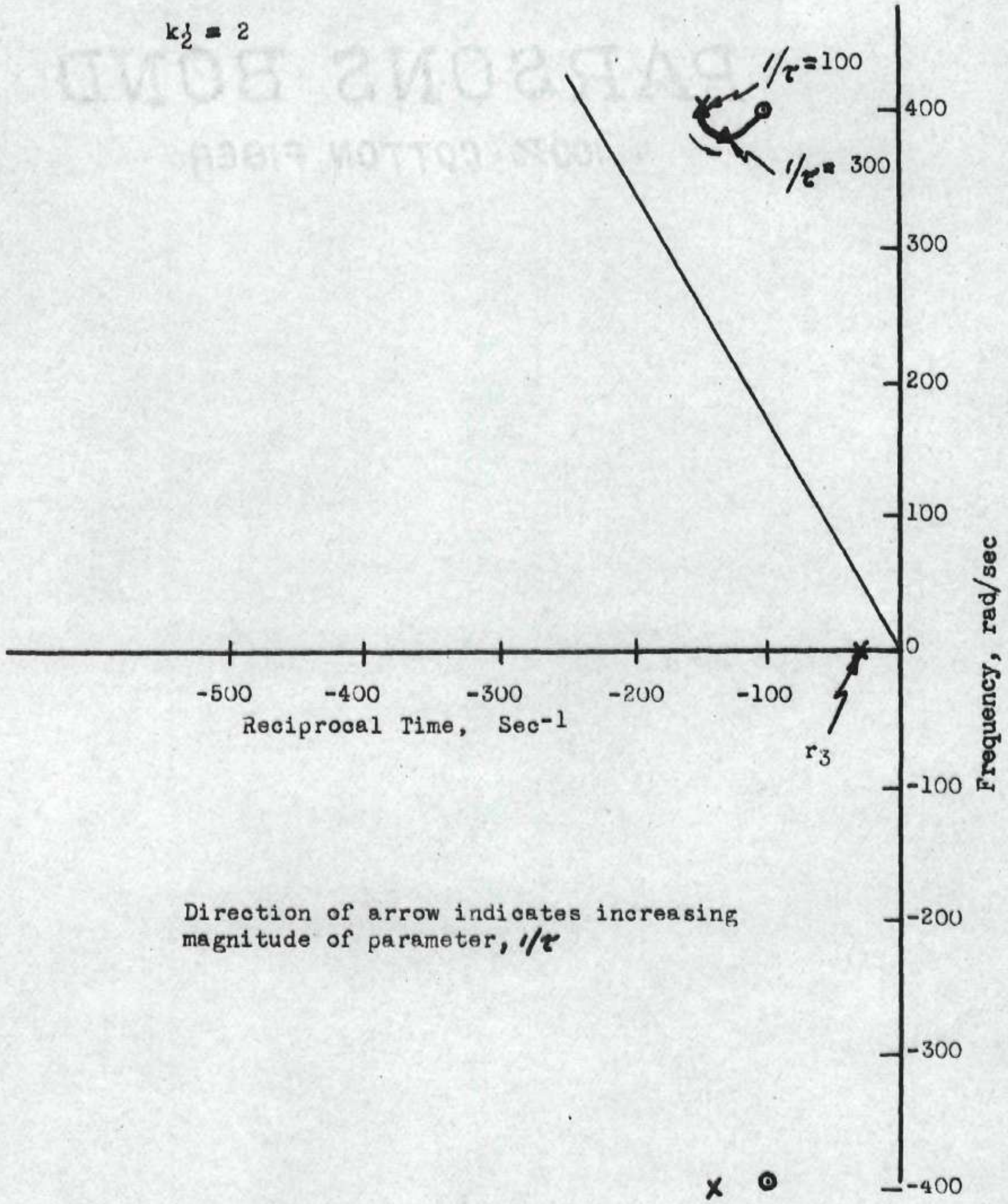


Figure 6, Test 2, $1/\tau$ Variable

$$GH(s) = \frac{1/\tau\,(s + 100 \pm 395.5j)}{(s + 120)(s + 272.6 \pm 345.5j)}$$

$k_2^1 = 4$

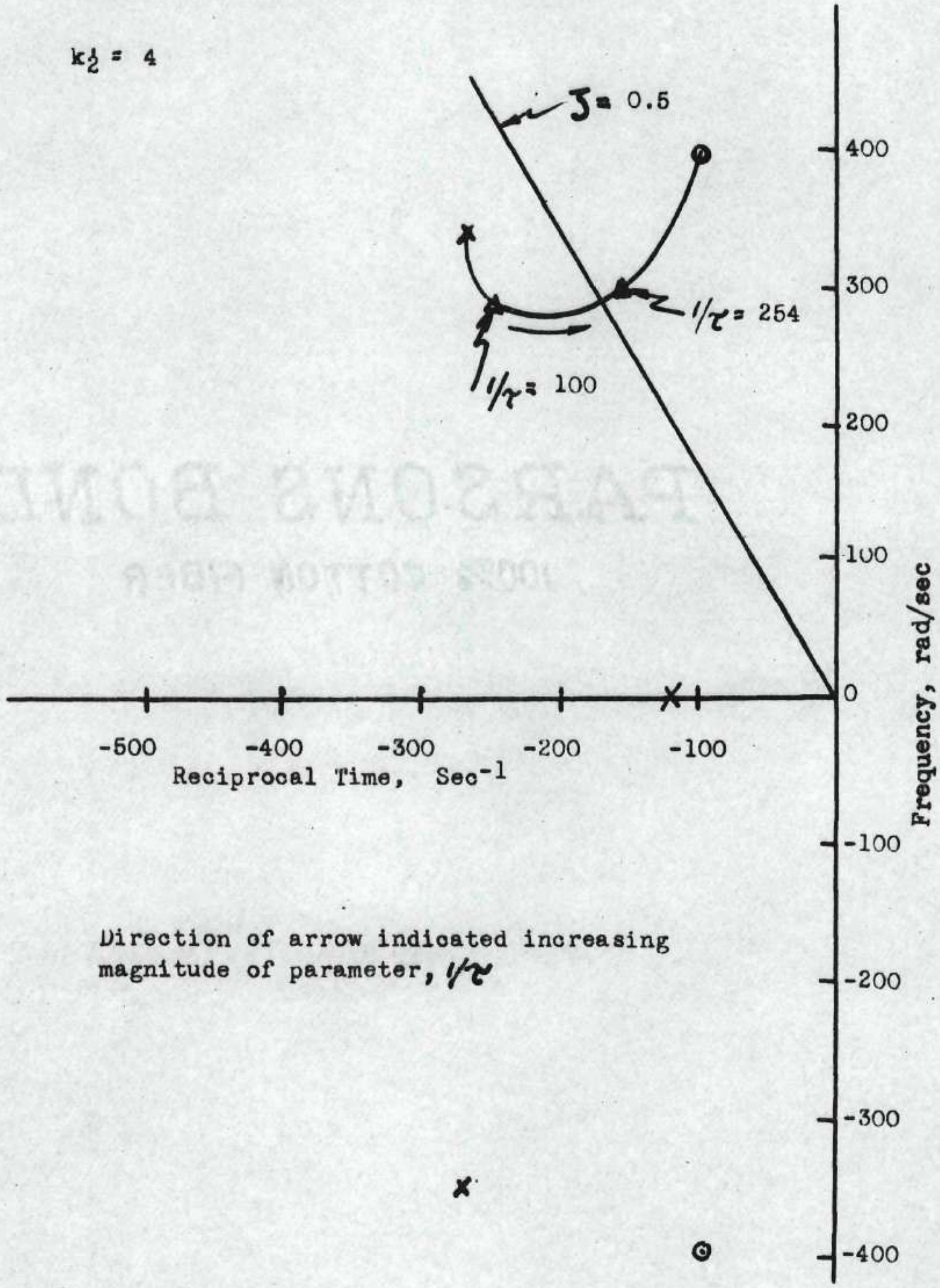


Figure 7, Test 3, $1/\tau$ Variable

$$GH(s) = \frac{116.3\, k_2' \,(s + 100 \pm 200j)}{(s + 100)(s + 100 \pm 395.5j)}$$

$1/\tau = 100$

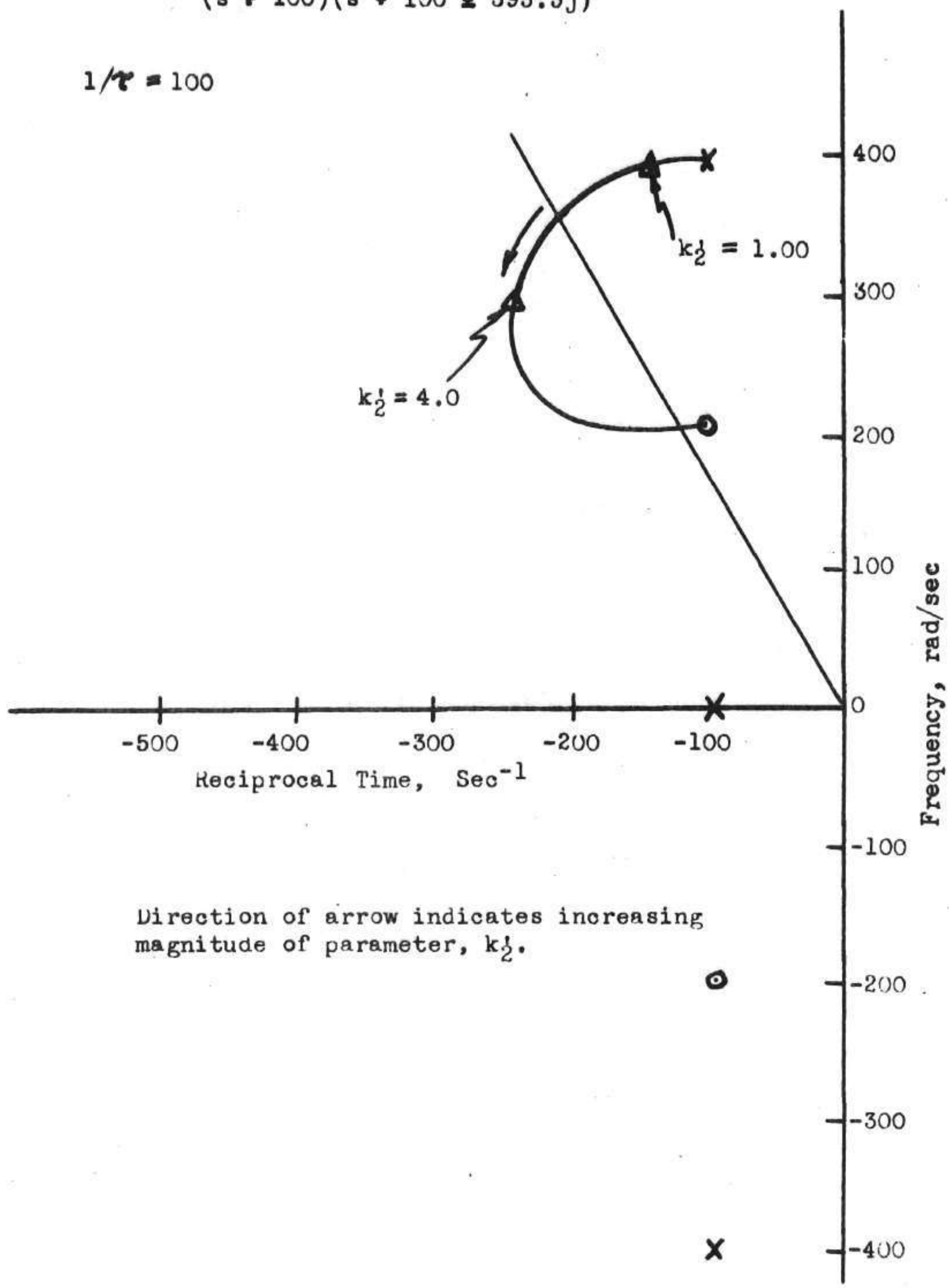


Figure 8, Test 4, k_2' Variable

$$GH(s) \quad \frac{116.3\, k_{\frac{1}{2}}\,(s + 100 \pm 200j)}{(s + 300)(s + 100 \pm 395.5j)}$$

$1/\tau = 300$

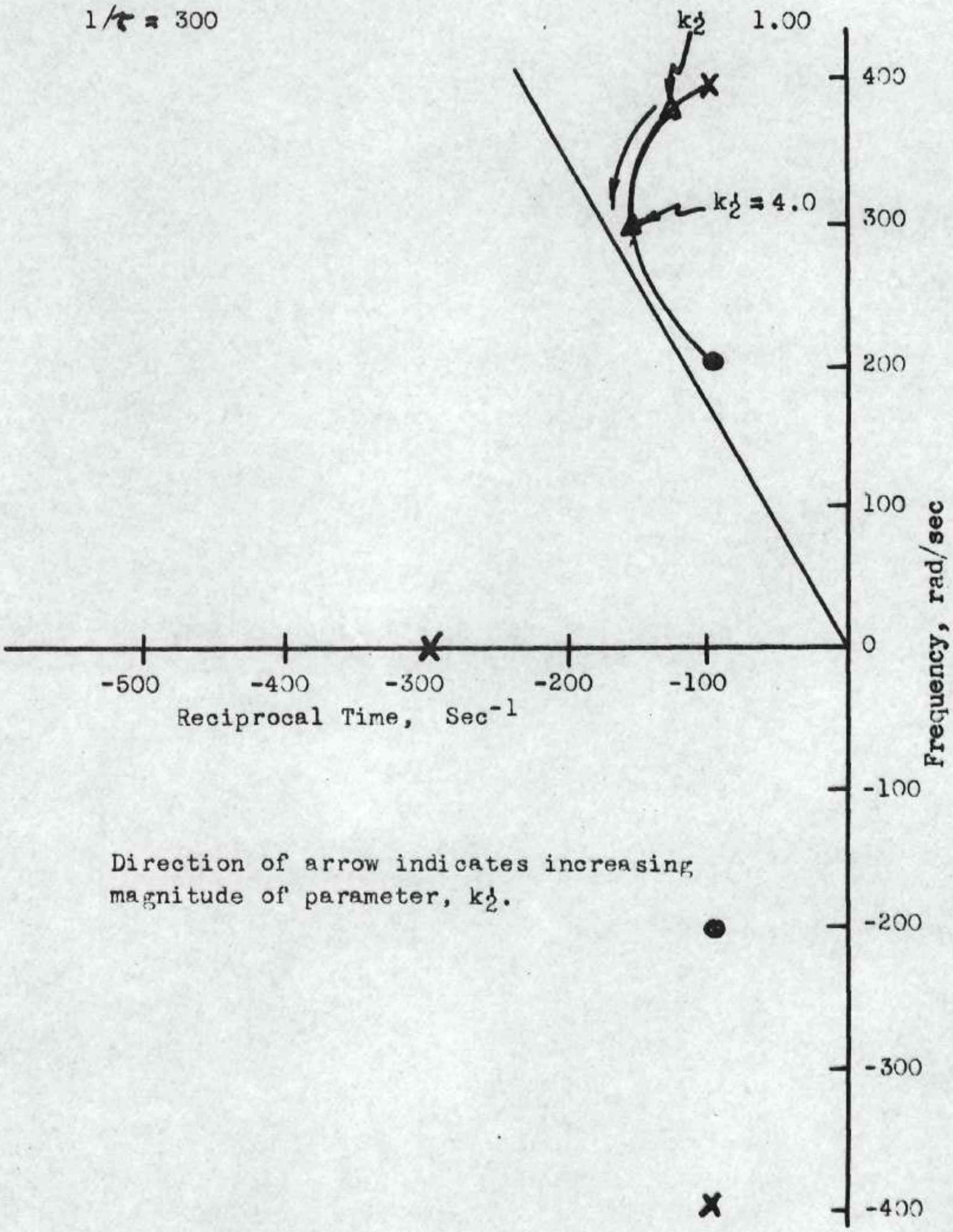
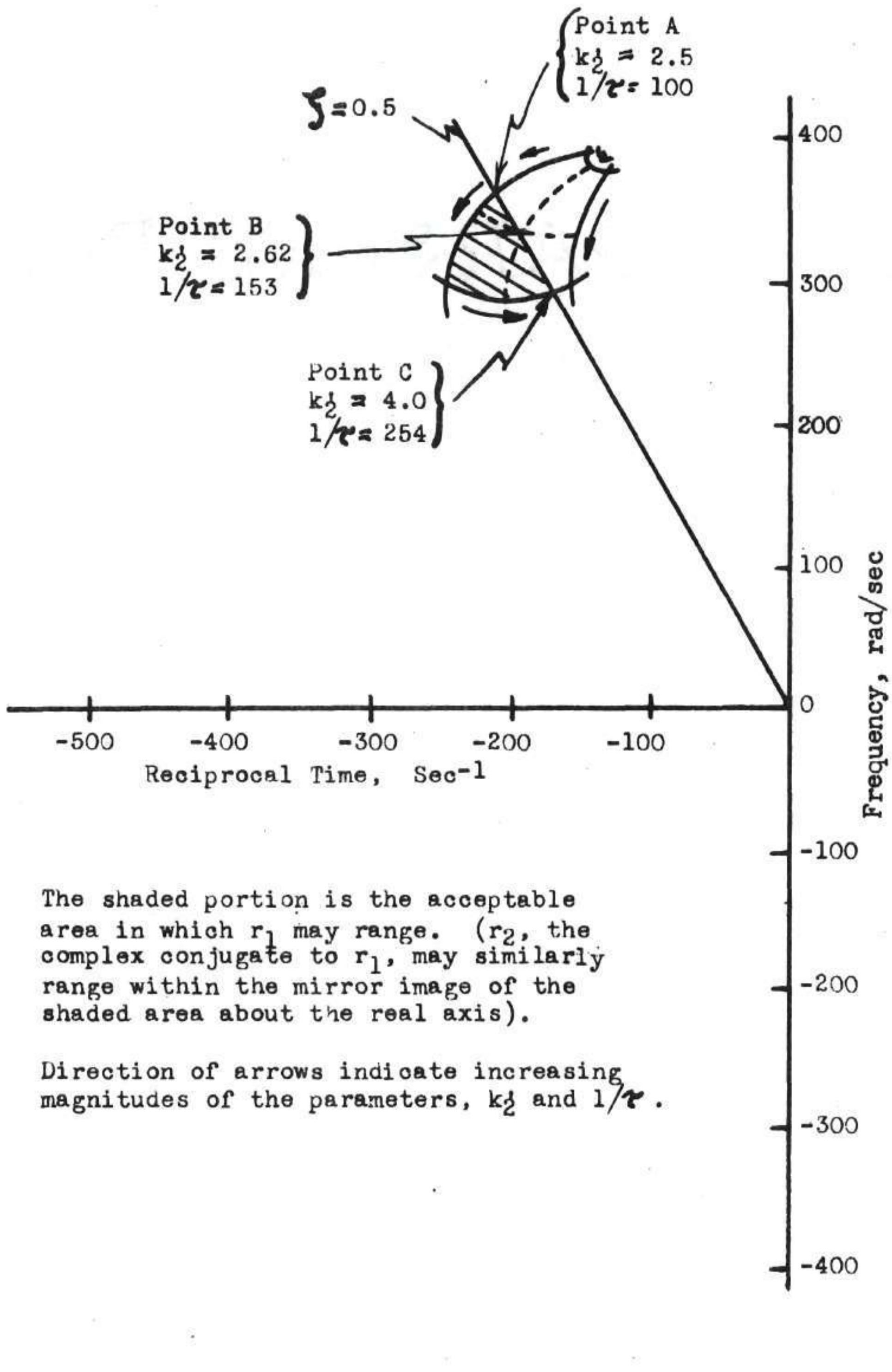


Figure 9, Test 5, $k_{\frac{1}{2}}$ Variable



The shaded portion is the acceptable area in which r_1 may range. (r_2, the complex conjugate to r_1, may similarly range within the mirror image of the shaded area about the real axis).

Direction of arrows indicate increasing magnitudes of the parameters, k_2 and $1/\tau$.

Figure 10, Test 6, Contour Plot, k_2 and $1/\tau$ Variable

$$P_i = 2500 \text{ lbs/in}^2$$

Substitution of these values into equation (68) yields,

$$V_t = 7000 \, (\tau k_2) \tag{73}$$

The smallest acceptable bottle size results when $k_2' = 4.00$ $\text{in}^5/\text{lb-sec}$ and $1/\tau = 254 \text{ sec}^{-1}$ taken from point C on the contour plot in Figure 10. These values were substituted into equation (73), which resulted into the following accumulator bottle size,

$$V_t = 110.3 \text{ in}^3 \tag{74}$$

Derivation of k_1'.--The calculation of k_1' was similar to that of the original k_1, except that now the plenum bottles, as well as chambers a and b, must be filled and compressed within the prescribed one second. Again, assuming the perfect gas law,

$$PV = w_a RT/g \tag{75}$$

where, V equals the combined volumes of chamber a (or b) plus plenum a (or b). All other terms were previously defined. Substituting values into equation (71), resulted in,

$$w_a = 1.527 \text{ lbs}, \tag{76}$$

If it is assumed, as before, that the flow rate is constant for a one second period, then,

$$\dot{w}_a = 1.527 \text{ lbs/sec} \tag{77}$$

From equation (53)

$$k_1' = 354 \text{ in}^2/\text{sec} \tag{78}$$

Derivation of K_s/C_s, Augmented System.--For the first time in this study, all knowns were substituted in the coefficients of s and both the oscillatory mode and the time response characteristics were studied concurrently. For convenience all the constants and computed parameters of the system except K_s/C_s, which was the remaining parameter to be derived, were rewritten below.

k_s/C_s, variable

$k_1' = 354 \text{ in}^2/\text{sec}$

$k_2' = 4 \text{ in}^5/\text{lb-sec}$

$k_3 = 8.6 \times 10^{-3}$

$1/\tau = 254 \text{ sec}^{-1}$

$V_t = 110 \text{ in}^3$

$m_L = 0.1 \text{ lb-sec}^2/\text{in}$

$C_L = 20 \text{ lb-sec/in}$

$K_L = 5000 \text{ lb/in}$

$A = 10 \text{ in}^2$

$G_1 = 3.0$

$G_2 = G_3 = 0.371$

Substituting the above values into the characteristic equation for the augmented system, the following coefficients of s were obtained,

$$GH(s) = \frac{K_s/C_s\ (s + 3.08)(s + 423.5)(s + 246.2 \pm 195.5j)}{s(s + 4.31)(s + 581)(s + 166.8 \pm 290.8j)}$$

K_s/C_s = 2.47 at the symbol ▲.

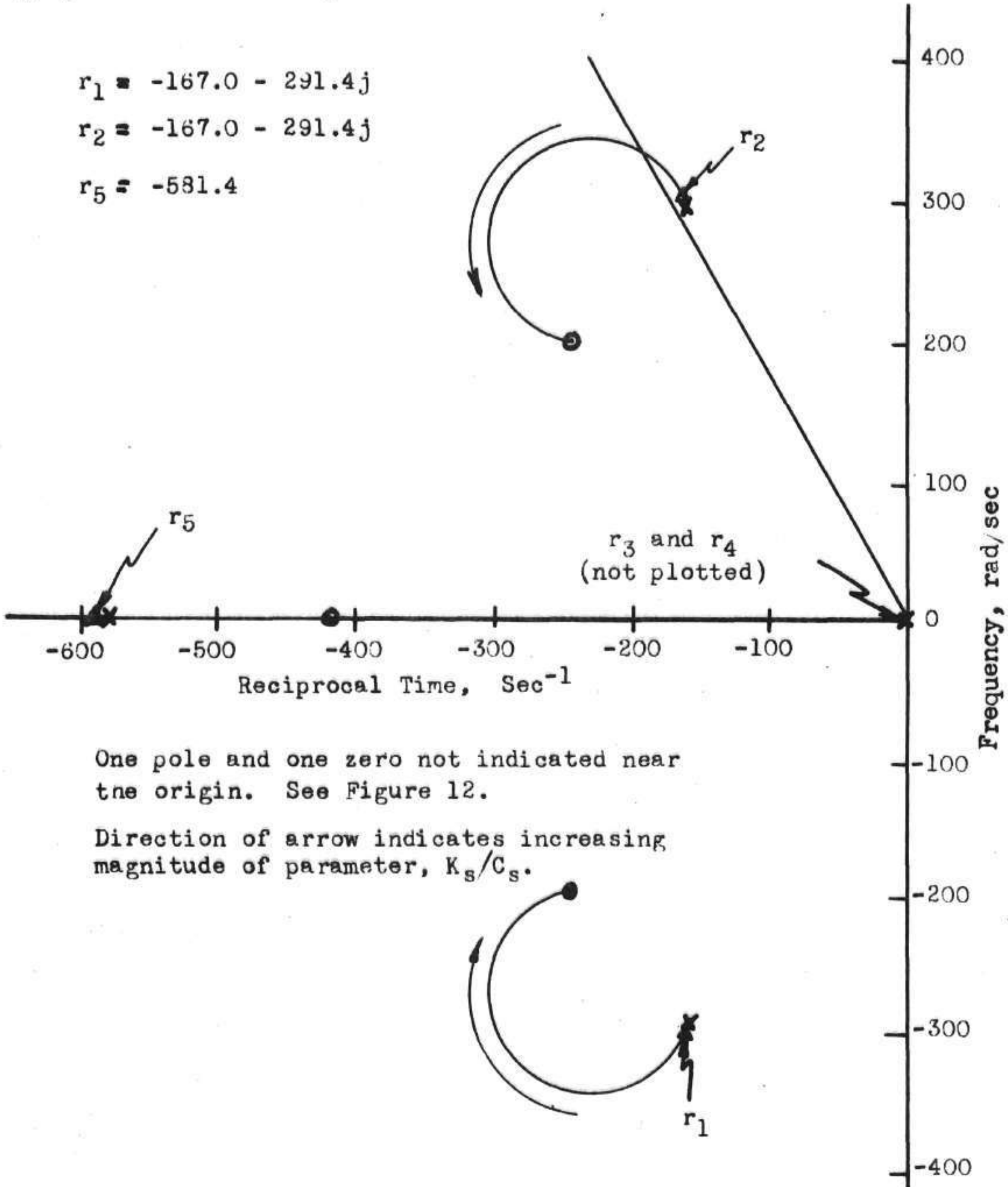


Figure 11, Test 7, K_s/C_s Variable, Oscillatory Mode

$$GH(s) = \frac{K_s/C_s\ (s + 3.08)(s + 423.6)(s + 246.2 \pm 195.5j)}{s(s + 4.31)(s + 581)(s + 166.8 \pm 290.8j)}$$

K_s/C_s = 2.47 at the symbol Δ.

r_1 = -1 (prescribed value)

r_4 = -4.90

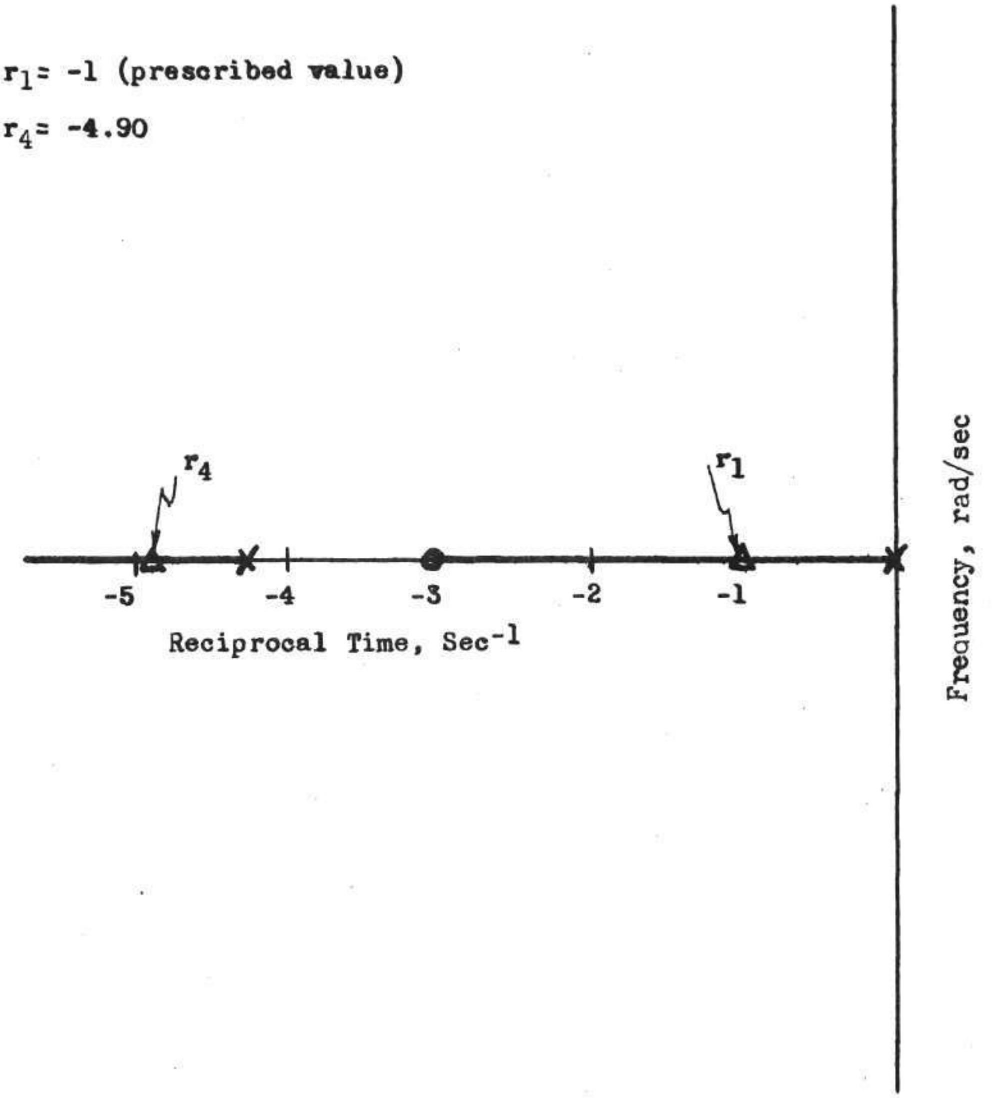


Figure 12, Test 8, K_s/C_s Variable, Real Axis

$$
\begin{array}{ll}
s^5 & 1 \\
s^4 & 919 + K_s/C_s \\
s^3 & 310{,}140 + 919\ K_s/C_s \\
s^2 & 66{,}576{,}000 + 310{,}140\ K_s/C_s \\
s^1 & 281{,}260{,}000 + 42{,}808{,}000\ K_s/C_s \\
s^0 & 129{,}100{,}000\ K_s/C_s
\end{array}
$$

The total $\Delta(s)$ function of the system evolved to the following,

$$
\begin{aligned}
\Delta(s) = \; & s(s + 4.31)(s + 581)(s + 166.8 \pm 290.8j) \\
& + K_s/C_s(s + 3.08)(s + 246.2 \pm 195.5j)
\end{aligned} \tag{79}
$$

The GH(s) function was obtained from equation (79) in the manner discussed in the Appendix.

$$
GH(s) = \frac{K_s \Delta C_s (s + 3.08)(s + 423.5)(s + 246.2 \pm 195.5j)}{s(s + 4.31)(s + 581)(s + 166.8 \pm 290.8j)} \tag{80}
$$

The GH(s) function was plotted in the root locus plots of Figures 11 and 12. Due to the large scale change, two figures for a single GH(s) function were used, for clarity in determining and reading the root values.

There are three real damping terms in the total system. The most important root of these terms is r_3. Therefore, r_3 was set equal to the specified value of one $\sec^{-1}$ (assumed equivalent to a one-second time constant). This value of r_3 defined the value of K_s/C_s, which in turn, defined the remaining roots. The time history in Figure 15 indicates the validity of assuming r_4 and r_5 negligible in their effect on the time response of the system.

With the final value of K_s/C_s, all roots were defined in Figures 11 and 12, and were tabulated below,

$$r_{1,2} = -167 \pm 291.4j$$

$$r_3 = -1 \quad \text{(according to specifications)}$$

$$r_4 = -4.90$$

$$r_5 = -581.4$$

Investigation of K_L.--All work up to this point has been done with K_L = 5000 lbs/in, the full load condition. Let the load now be variable to determine the root variation, particularly the largest real root and the complex pair as $0 \leq K_L \leq 5000$.

All knowns were substituted into the coefficients of s, with K_L variable.

K_L variable

$$1/\tau = 254$$

$$k_1' = 354$$

$$k_2' = 4$$

$$k_3 = 8.6 \times 10^{-3}$$

$$m_L = 0.1$$

$$C_L = 20$$

$$K_s/C_s = 2.47$$

$$A = 10$$

$$G_1 = 3$$

$$G_2 = G_3 = 0.371 .$$

This resulted in, the following coefficients of s,

$$
\begin{array}{ll}
s^5 & 1 \\
s^4 & 921.67 \\
s^3 & 262{,}433 + 10(K_L) \\
s^2 & 31{,}225{,}000 + 7216.7\,(K_L) \\
s^1 & 332{,}957{,}000 + 17{,}765\,(K_L) \\
s^0 & 319{,}300{,}000.
\end{array}
$$

The above constants were substituted into the characteristic equation,

$$
\begin{aligned}
\Delta(s) \;=\; & 10(K_L)(s^3 + 721.67s^2 + 1776.5s) \\
& + (s^5 + 921.67s^4 + 262{,}433s^3 + 31{,}225{,}000s^2 \\
& + 332{,}957{,}000s + 319{,}300{,}000)
\end{aligned}
\tag{81}
$$

The GH(s) function was reduced from equation (81) to the following in the manner discussed in the Appendix.

$$
GH(s) = \frac{10(K_L)s(s + 2.47)(s + 719.18)}{(s + 1.063)(s + 10.62)(s + 541.3)(s + 184.3 \pm 134.5j)} \tag{82}
$$

The GH(s) function was plotted in the root locus plots of Figures 13 and 14. The root shift due to $0 \leq K_L \leq 5000$ ranged from $\zeta = 0.8$ to $\zeta = 0.5$ in the oscillatory mode, which was satisfactory according to the specifications. The negative real root under no-load conditions ($K_L = 0$) shifted to $r_1 = 1.06$. This means the damping term changed to $1/r_1 = 1/1.06 = 0.94$ sec., resulting in a slightly faster response.

$$GH(s) = \frac{10\,K_L\,s(s+2.47)(s+719.18)}{(s+1.063)(s+10.62)(s+541.3)(s+184.3 \pm 134.5j)}$$

$1/\tau = 254$
$k_1' = 354$
$k_2' = 4.0$
$K_s/C_s = 2.47$

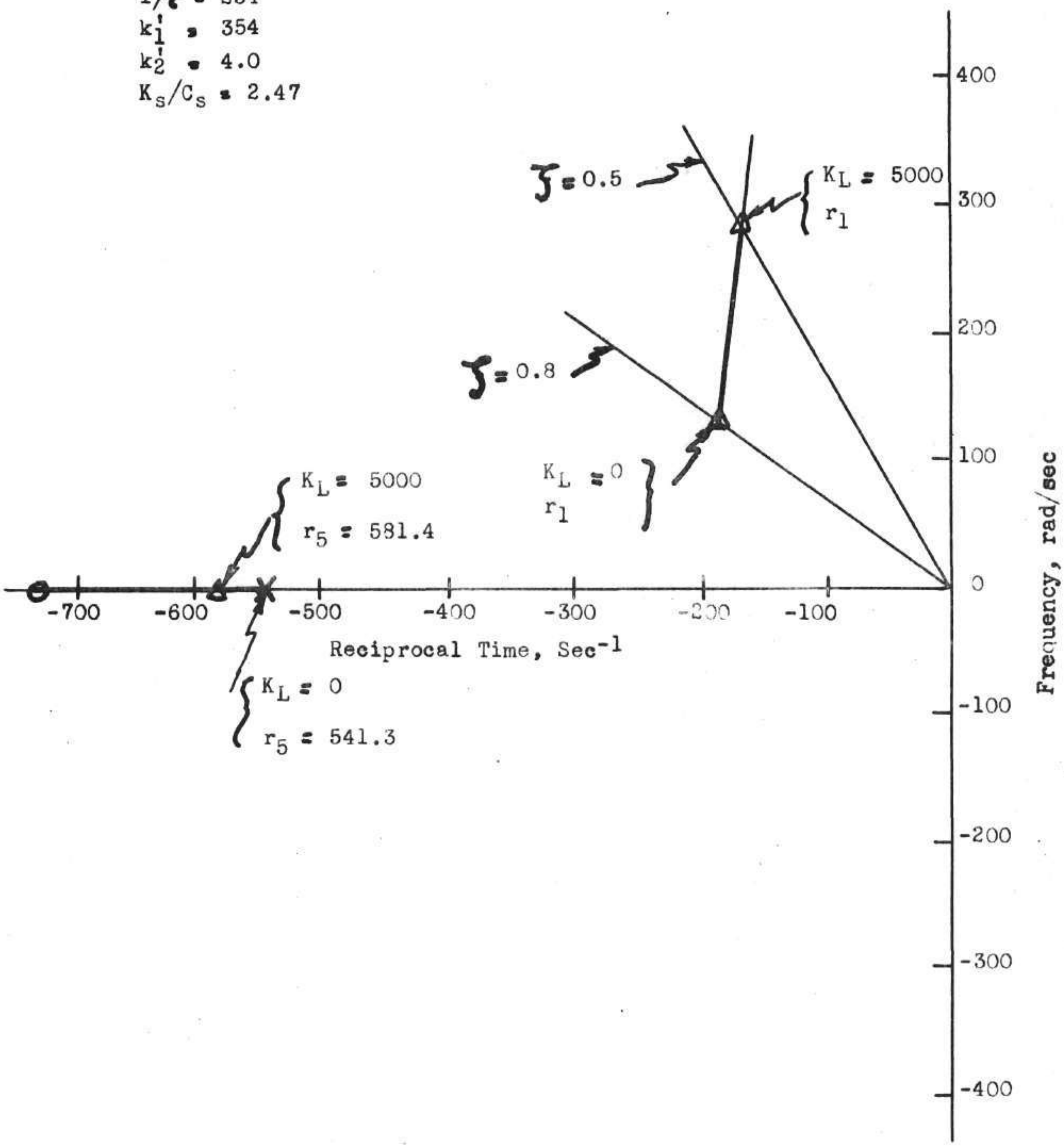


Figure 13. K_L Variable, Oscillatory Mode, Test 9.

$$GH(s) = \frac{10\,K_L\,s(s + 2.47)(s + 719.18)}{(s + 1.063)(s + 10.62)(s + 541.3)(s + 184.3 \pm 134.5j)}$$

$1/\tau = 254$

$k_1' = 354$

$k_2' = 4.0$

$K_s/C_s = 2.47$

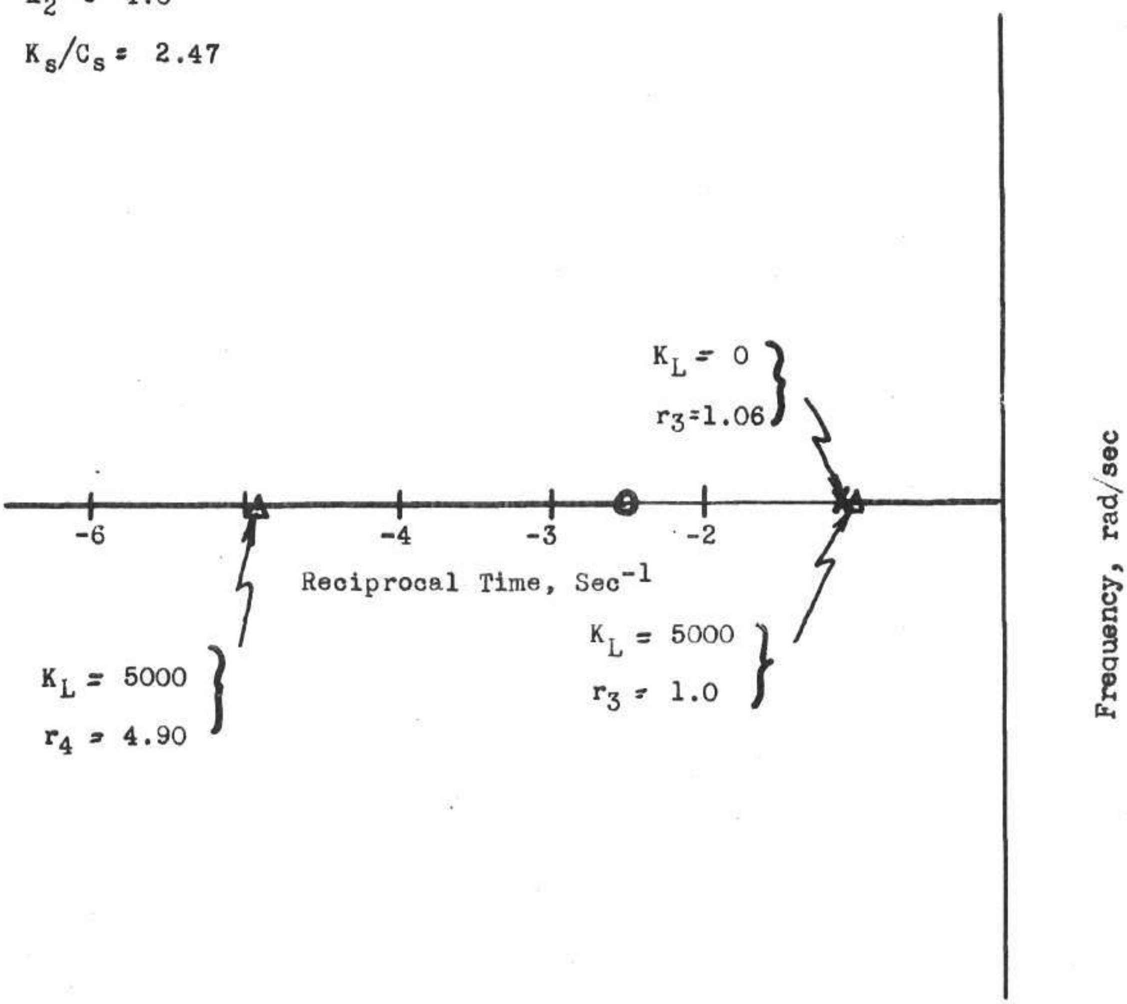


Figure 14, Test 10, K_L Variable, Real Axis

<u>Time History of System due to Step Input to $x_i(s)$</u>.--The transient time history was computed here only for academic reasons since, from a stability and control standpoint, the analysis is actually completed. The time history was included to show the relationship between the roots of $\Delta(s)$ and the transient behavior of the system.

Equation (28) was extended to the augmented system by substituting k_2 with the expression $k_2's(1/\tau + s)$, as was done in previous sections, to the following,

$$x_o(s) = \frac{Ak_1 b\, x_i(s)/(a + b)m_L k_3\, (s + K_s/C_s)(s + 1/\tau)}{\Delta(s)} \tag{83}$$

where,

$$\begin{aligned}\Delta(s) = {} & Ak_1(K_s + sC_s)(1 + \tau s)G_3/G_1 + Ak_1C_s(1 + \tau s)sG_2/G_1 \\ & + (K_s + sC_s)\, A^2(1 + \tau s)s \\ & + (m_Ls^2 + C_Ls + K_L)(k_2's + sk_3 + s^2k_1)\end{aligned} \tag{84}$$

And now, from the root locus plots,

$$\Delta(s) = (s + r_1)(s + r_2)(s + r_3)(s + r_4)(s + r_5) \tag{85}$$

where the roots are listed on page 45 above.

Let a step input, $x_i(s) = |x_i|/s$ and $\Delta(s)$ from equation (85) be substituted into equation (83). Then,

$$x_o(s) = \frac{Ak\, b/(a + b)m_Lk_3\, |x_i|\, (s + K_s/C_s)(s + 1/\tau)}{s(s + r_1)(s + r_2)(s + r_3)(s + r_4)(s + r_5)} \tag{86}$$

For an immediate check on the compatibility of all constants, assumed and computed, the Final Value Theorem in equation (31) was used,

$$\lim_{t \to \infty} X_o(t) = \lim_{s \to 0} sx_o(s) = \frac{Ak_1 b \quad |x_i|(K_s/C_s)(1/\tau)}{(a+b)m_L k_3 \; r_1 r_2 r_3 r_4 r_5}$$

Substituting $|x_i| = 1.236$ (maximum travel) and all the other constants assumed and computed, then,

$$|x_o|_{max} = \frac{(10)(354)(3) \quad (1.236)(2.47)(254)}{(6)(0.1)(8.6 \times 10^{-3}) \; (112.800)(1)(4.90)(581.4)}$$

$$|x_o|_{max} = 5.0 \text{ inches}$$

This result is consistent with the original stipulation that for a full over input, the ram will move full over.

The method of partial fractions was used to return equation (81) to the time domain. This part of the work is not reported since it was lengthy and also was not a part of the objective of this study. The time history equation evolved to the following,

$$x_o(t) = |x_i| \; 4.04 - 0.04\, e^{-167.0t} \cos(291.4t + 67°) \quad (87)$$
$$-(3.01e^{-t} + 1.00\, e^{-4.90t} + 0.01\, e^{-581.4t})$$

Equation (87) was plotted in Figure 15 to indicate the system's unique time history caused by a particular input. The system's oscillatory mode could not be plotted since it was of such low amplitude. However, the high frequency mode was present and, as the transfer function indicated, would have caused an unstable condition had it not been sufficiently damped.

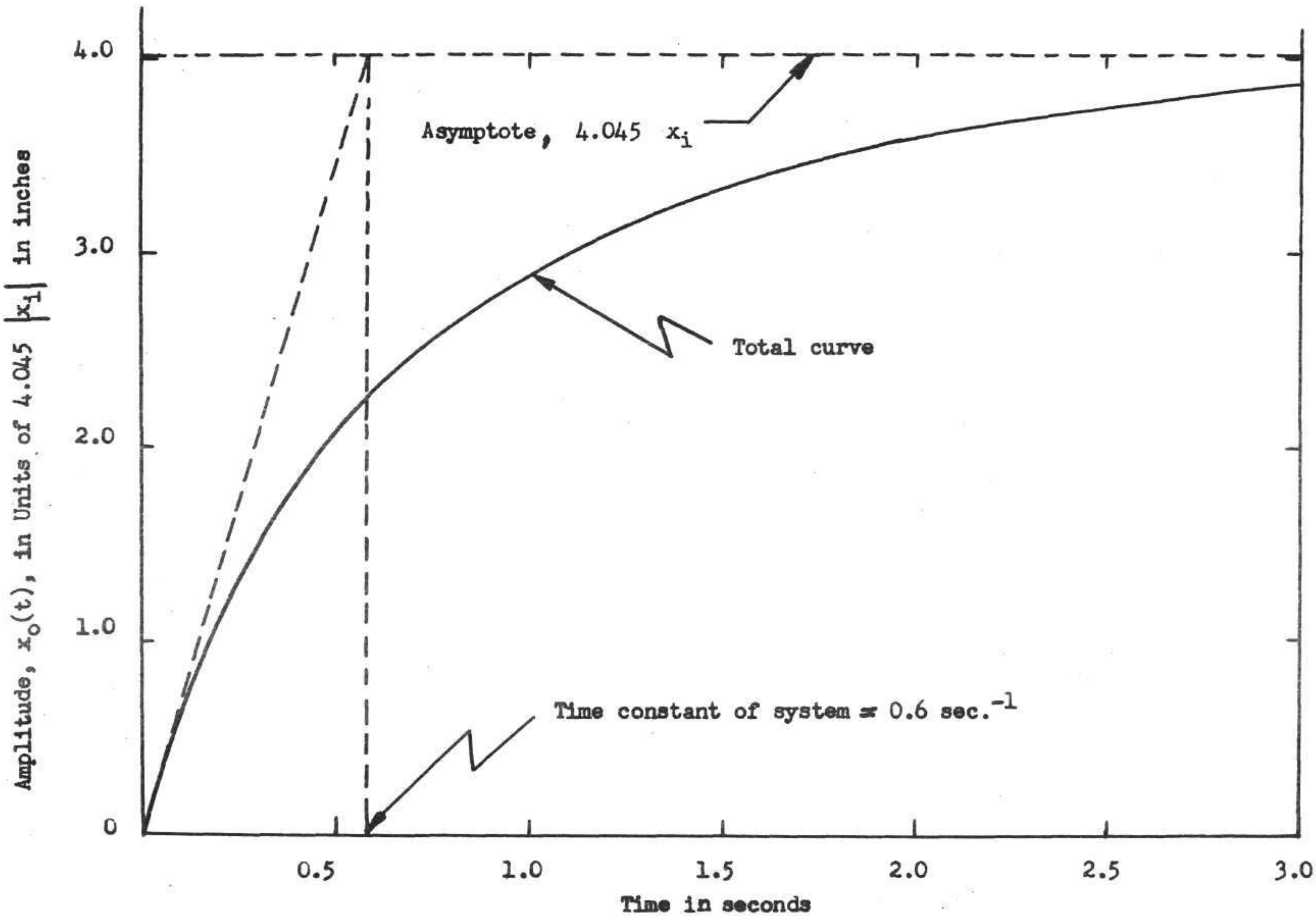


Figure 15, Time History of $x_o(t)$ for Input $|x_i|$

CHAPTER IV

DISCUSSION OF RESULTS

Discussion.--Two configurations of the system were studied. The original system contained no pressure-transient-feedback. The second or augmented system included the external damping which consisted of plenum chambers and the connecting passages.

The first configuration was analyzed by the root locus method and plotted in Figure 5. The graph indicated the system was too lightly damped and, hence, was unacceptable. This condition might have been corrected by one of several methods, including an external spring and dashpot, or leakage across the ram head, or by the method suggested in this report. The plenum bottle configuration, or augmented system causes the least loss of energy for actuation.

The augmented system increased the complexity of the system from fourth to fifth order. It also increased the weight rate of flow requirements, which meant the valve constant k_1 had to be recomputed.

The augmented system was analyzed by optimizing the oscillatory mode first, in which k_2' and $1/\tau$ were determined. Secondly, the total system was studied, with particular emphasis on the response characteristics, in which K_s/C_s was determined.

The finalized values for k_2' and $1/\tau$ defined the plenum bottle size. These two parameters entered into the calculations of the flow passage dimensions, but in such a way that there is no unique solution.

The assumption that r_4 and r_5 were such that they may be neglected, may indeed be subject to rebuttal. The time history in Figure 15 indicated the time constant is more nearly 0.6 seconds rather than the prescribed one second. However, r_3 is the most dominant real root and its location in the complex plane can be shifted to the left (see Figure 12) by increasing K_s/C_s only. However, to recompute the time history curve, all roots would have to be recomputed. From a practical standpoint, all requirements have been satisfied and, if a more precise response were needed, the system would normally be tested in mock up. This work has served its purpose in determining the limited area for further refinement in the laboratory or on the actual hardware.

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS

The addition of the velocity limiting feedback, in which K_s/C_s is the defining parameter, was done in order to limit the velocity of the ram to the prescribed value irrespective of load. Thus, a system could be designed for a full load without limiting the velocity but would probably cause self-destruction under a no-load condition. The addition of velocity limiting introduced non-linearities whose magnitudes were dependent on the magnitude of K_s and C_s and on the system's coulomb friction, particularly that of the valve. Throughout this study a rigid input was implied. Hence, any change or displacement to input could be absorbed completely or partially by the spring and damper. This point was touched upon in the section "Determination of K_s/C_s" in Chapter III. It was stated that a 5% displacement on x_i would be just sufficient to overcome the reaction force due to the valve coulomb friction. Therefore, the deleterious effect was reduced to an acceptably low level by setting K_s and C_s sufficiently high to the point where a 5% displacement on x_i would initiate a change in the valve displacement. These values can be increased arbitrarily high, so long as K_s/C_s remain unchanged but there is a point where structural difficulties arise. Also it is possible under these conditions that the reaction forces transmitted back to the input could cause unwanted displacements. However, this would indicate either that the input was not sufficiently rigid or that the coulomb frictional forces were extremely high. In this event, the system would probably not function and the analysis technique would have no meaning.

The addition of plenum bottles is a flexible and efficient way of increasing damping. It has the advantage of having no moving parts. However, since the passages are thin they may become clogged if the air used is not clean of dust particles. The capillary passages must be precision made and could be expensive to fabricate, which means there are considerations to weigh other than dynamic ones when deciding the advisability of using plenum bottles. If the system is to be used only for a short time, say for a space vehicle which is in the atmosphere only a few seconds, then straight leakage across the ram head may actually be more efficient. That is to say, the same k_2' may be accomplished in one of many ways, but this study shows that some type of external damping must be added to the system.

The final time history curve in Figure 15 shows that the rise time is somewhat faster than the required $1/r_3 = 1$ second. In the calculation of k_1', the valve characteristic for the augmented system, it was assumed that the weight rate of flow was $\dot{w}_a = 0.764$ lbs/sec. The fact that this value may have been too high has been previously discussed. Another factor, the neglect of the real roots r_4 and r_5, did increase the final response time.

It should be stated that a response which is slightly too fast can be correlated by adjusting the parameter K_s/C_s. This might entail a complete reappraisal of the analysis or simply a screwdriver adjustment.

Only three parameters were found in this study by the root locus method, k_2', $1/\tau$, and K_s/C_s and the system was only fifth order. This technique is equally applicable to more complicated systems, but it may be advisable to use a digital computer.

There are no direct experimental data to support the results of this study. However, the analysis technique is similar to that the author used in the analytical dynamic portion of the report entitled "Design Development and Functional Test of a 1000° F Pneumatic System, Phase II," by Lockheed Aircraft Corporation, Georgia Division. The results of the Lockheed system were completely tested by a full-scale mock up in the laboratory, and all parameters determined by the root locus method were found sufficiently optimized to cause the system to behave, within narrow limits, in the prescribed manner.

Let it then be assumed that since one system was designed by the method presented in this study and tested satisfactorily, the other system, after a similar design study, would also behave in the prescribed manner.

For security reasons the postulated system is different in pressures, application, valve characteristics, and in all dimensions from the original Lockheed system.

APPENDIX

APPENDIX

ROOT LOCUS METHOD

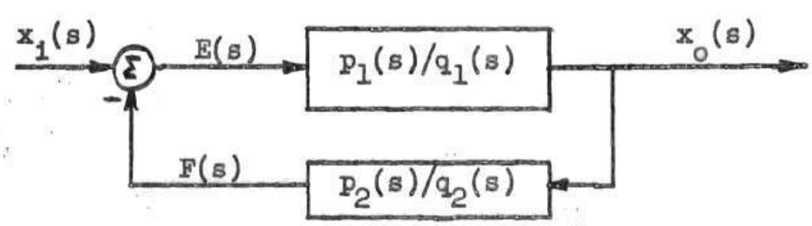
The dynamic analysis of the postulated system was accomplished by the root locus method. Again, as in the development in Chapter III, the development was presented principally by example rather than in general terms. The presentation here contains more detail in the development of the three most used functions, the characteristic equation, transfer function, and the GH(s) function.

The Characteristic Equation.--Let the characteristic equation be called $\Delta(s)$. The roots of $\Delta(s)$ are the exponents of the base e in the final time history, and therefore determine the natural frequencies and response times of the system. The characteristic equation is the denominator of the transfer function.

The Overall Transfer Function.--The mathematical definition of the overall transfer function of the system is the output divided by the input, $x_o(s)/x_i(s)$.

The GH(s) Function.--The definition of this function was deferred to the section below on the mathematical development where it is defined in the development of the transfer function and $\Delta(s)$.

Mathematical Development.--Before the selected Test 4 is introduced, let the following simplified block diagram be considered,



In any linear system, the transfer function of a given component is the output divided by the input. Thus, the transfer function, $x_o(s)/E(s) = p_1(s)/q_2(s)$, and similarly, $F(s)/x_o(s) = p_2(s)/q_2(s)$. We seek the transfer function, $x_o(s)/x_i(s)$, which is the transfer function of the overall system.

The following equations were taken from the diagram above,

$$x_i(s) - F(s) = E(s) \tag{88}$$

$$F(s) = x_o\left[p_2(s)/q_2(s)\right] \tag{89}$$

$$E(s) = x_o\left[q_1(s)/p_1(s)\right] \tag{90}$$

Equations (89) and (90) were substituted into equation (88) which resulted in the following.

$$x_o(s)/x_i(s) = \frac{p_1(s)/q_1(s)}{1 + p_1p_2(s)/q_1q_2(s)} \tag{91}$$

The denominator of equation (91), $\Delta(s)$ was equated to zero to find the roots of the system. Thus,

$$\Delta(s) = 1 + p_1p_2(s)/q_1q_2(s) = 0 \tag{92}$$

The GH(s) function is defined by the term, $p_1p_2(s)/q_1q_2(s)$. The development delineated above is used by practically all designers, and as such contains no new or original material.

However, the manner in which the GH(s) function is used marks the important difference between the several analytical techniques. The primary purpose of this report is to illustrate a new approach to stability analysis by introducing a new method of treating this fundamental function.

Equation (92) was rearranged to the following,

$$1 + GH(s) = 0 \tag{93}$$

$$GH(s) = -1$$

$$GH(s) = 1e^{i\pi(1 \pm 2n)} \tag{94}$$

where

$$n = 0, 1, 2, 3 \ldots$$

Equation (94) implies that two conditions must be satisfied, and these are,

Condition 1,

$$|GH(s)| = 1 = |p_1p_2(s)/q_1q_2(s)| \tag{95}$$

and,

Condition 2,

All angles in the GH(s) function, which contains many complex numbers, must total 180°. Both conditions are further clarified below and in Figure 16.

The root locus method is a simplified method of determining the zeroes of $p_1p_2(s) + q_1q_2(s) = 0$ by computing the zeroes of $p_1p_2(s)$ and zeroes of $q_1q_2(s)$ individually. Stated briefly, it is the interim method of arriving at the following,

$$\Delta(s) = p_1p_2(s) + q_1q_2(s) = (s + r_1)(s + r_2) \ldots (s + r_n) = 0 \tag{96}$$

Where $r_1, r_2, \ldots r_n$, which may be complex and which are functions of the systems parameters, are the exponents of the base e in the time domain.

The roots of $\Delta(s) = 0$ determine the time behavior of the system. Thus for every set of parameter values there exists a unique time response (for a given input). The objective of this work is to find a set of parameter values that are both physically realizable and that will allow the time history to fit the prescribed requirements. The prescribed requirements in this study are that the system's oscillatory mode have a damping ratio, $\zeta = 0.5$ and that the principal time constant be $1/r_3 = 1.0$ sec. The roots with the subscripts were defined in the List of Symbols.

Illustrative Example.--Test 4, taken from Chapter III, was used to develop, or extend, the above general treatment into a particular example.

The denominator, $\Delta(s)$, was expanded again for Test 4 to show explicitly the way the two functions $\Delta(s)$ and GH(s) were derived, how they were related, and how they are related to the root locus plot.

The applicable coefficients of s were selected from the section "Oscillatory Mode Investigation k_2' and $1/\tau$ Variable," page 28. The value $1/\tau = 100$ was substituted into the coefficients with k_2 the single remaining varying parameter as follows,

$$\begin{aligned} \Delta(s) = 0 = s^3 &+ \left[k_2'(116.3) + 200 + 100\right]s^2 \\ &+ \left[k_2'(23{,}260) + 166{,}300 + 200(100)\right]s \\ &+ \left[k_2'(5{,}815{,}000) + 166{,}300(100)\right] \end{aligned} \tag{97}$$

Equation (93) was rearranged to the following,

$$\Delta(s) = (s^3 + 300s^2 + 186{,}300s + 16{,}630{,}000) + k_2'(116.3s^2 + 23{,}260s + 5{,}815{,}000) \tag{98}$$

The quadratic and cubic terms in equation (98) were broken up, as follows,

$$\Delta(s) = 116.3\ k_2'(s + 100 \pm 200j) + (s + 100)(s + 100 \pm 395.5j) \tag{99}$$

We seek the GH(s) function indicated in equation (94). Therefore both sides of equation (99) were divided by the expression $(s + 100)(s + 100 \pm 395.5j)$,

$$0 = \frac{116.3\ k_2'(s + 100 \pm 200j)}{(s + 100)(s + 100 \pm 395.5j)} - 1 \tag{100}$$

which resulted in the required form,

$$GH(s) = \frac{116.3\ k_2'(s + 100 \pm 200j)}{(s + 100)(s + 100 \pm 395.5j)} \tag{101}$$

Equation (101) is the general form of the GH(s) function for Test 4 in Figure 8.

It should be emphasized that the following analytical procedure is not the procedure used in this study. In the general development, the loci, and the root location on the loci, were found by graphic means. The work below tacitly assumes the root location and checks its accuracy by complex variable theory. To find the roots by mathematical means directly would require the use of a digital computer. Incidentally, in more complicated systems a digital computer would be recommended.

The work below serves two purposes: to check the graphic solution found in Test 4, and more important, to illustrate the method underlying the root locus technique.

In this application, the root selected was read from Figure 8 in the second quadrant (see Figure 16).

$$r_1 = -205 + 357j \tag{102}$$

For this particular root,

$$s = r_1 \tag{103}$$

Equations (102) and (103) were substituted into equation (101) which resulted in,

$$GH(s) = \frac{116.3\ k_2'(-105 + 557j)(-105 + 157j)}{(-105 + 357j)(-105 + 752.5j)(-105 - 38.5j)} \tag{104}$$

From equation (101)

$$GH(s) = 1\ e^{j\pi(1 + 2n)} \tag{105}$$

Equation (105) was substituted into equation (104) and rearranged to the following,

$$116.3\ k_2' \frac{AB}{CDE}\ e^{j(\phi_1+\phi_2+\phi_3+\phi_4+\phi_5)} = 1\ e^{j\pi(1 + 2n)} \tag{106}$$

where, A, B, C, D, and E are defined below.

Equations (102) implies two conditions:

Condition I, $(\phi_1 + \phi_2 + \phi_3 + \phi_4 + \phi_5) = (1 + 2n)\pi$

Condition II, $116.3\ k_2'\ AB/CDE = 1$

Condition I defines the loci, and Condition II defines the root locations for a particular k_2'.

$$\phi_1 = \tan^{-1} \frac{557}{-105} = \tan^{-1} -5.3048 = 180° - 79.3° = 100.7°$$

$$\phi_2 = \tan^{-1} \frac{157}{-105} = \tan^{-1} -1.6522 = 180° - 58.6° = 121.4°$$

$$\phi_3 = \tan^{-1} \frac{357}{-105} = \tan^{-1} -3.7570 = 180° + 75.1° = 255.1°$$

$$\phi_4 = \tan^{-1} \frac{752.6}{-105} = \tan^{-1} -7.9192 = 180° + 82.6° = 262.6°$$

$$\phi_5 = \tan^{-1} \frac{-38.5}{-105} = \tan^{-1} 0.3667 = 180° - 20.2° = \frac{159.8°}{899.6}$$

When, $n = 2$, $\phi = 899.6° - 2(360°) = 179.6°$

Condition 1 required $\phi = 180°$, and for practical purposes, is fulfilled.

$$A = \sqrt{105^2 + 557^2} = 566.8$$

$$B = \sqrt{105^2 + 157^2} = 188.9$$

$$C = \sqrt{105^2 + 357^2} = 372.1$$

$$D = \sqrt{105^2 + 752.5^2} = 759.8$$

$$D = \sqrt{105^2 + 38.5^2} = 111.8$$

Substituting the above values into Condition II, yielded,

$$\frac{116.3k_2' \, (566.8)(188.9)}{(372.1)(759.8)(111.8)} = 1 \qquad (107)$$

The value of k_2', computed from the above equation resulted in,

$$k_2' = 2.53 \quad \text{(and from Test 4, } k_2' = 2.55\text{)} \tag{108}$$

For every k_2' along the loci, there exists a unique set of roots, which in turn, defines a unique time history. Adjusting all the parameters so that they cause the time history to fit the prescribed requirements then becomes the final object of this study.

Oscillatory Root of Test 4

$r_1 = -205 + 357j$

Condition I,

$$\phi_1 = 100.7^\circ$$
$$\phi_2 = 121.4^\circ$$
$$\phi_3 = 255.1^\circ$$
$$\phi_4 = 262.6^\circ$$
$$\phi_5 = 159.8^\circ$$
$$\overline{899.6^\circ}$$

When $n = 2$, $\phi = 899.6^\circ - 2(360^\circ) = 179.6^\circ$

Condition II,

$$\frac{116.3\, k_2' \,(566.8)(188.9)}{(372.1)(759.8)(111.8)} = 1$$

when $k_2' = 2.53$

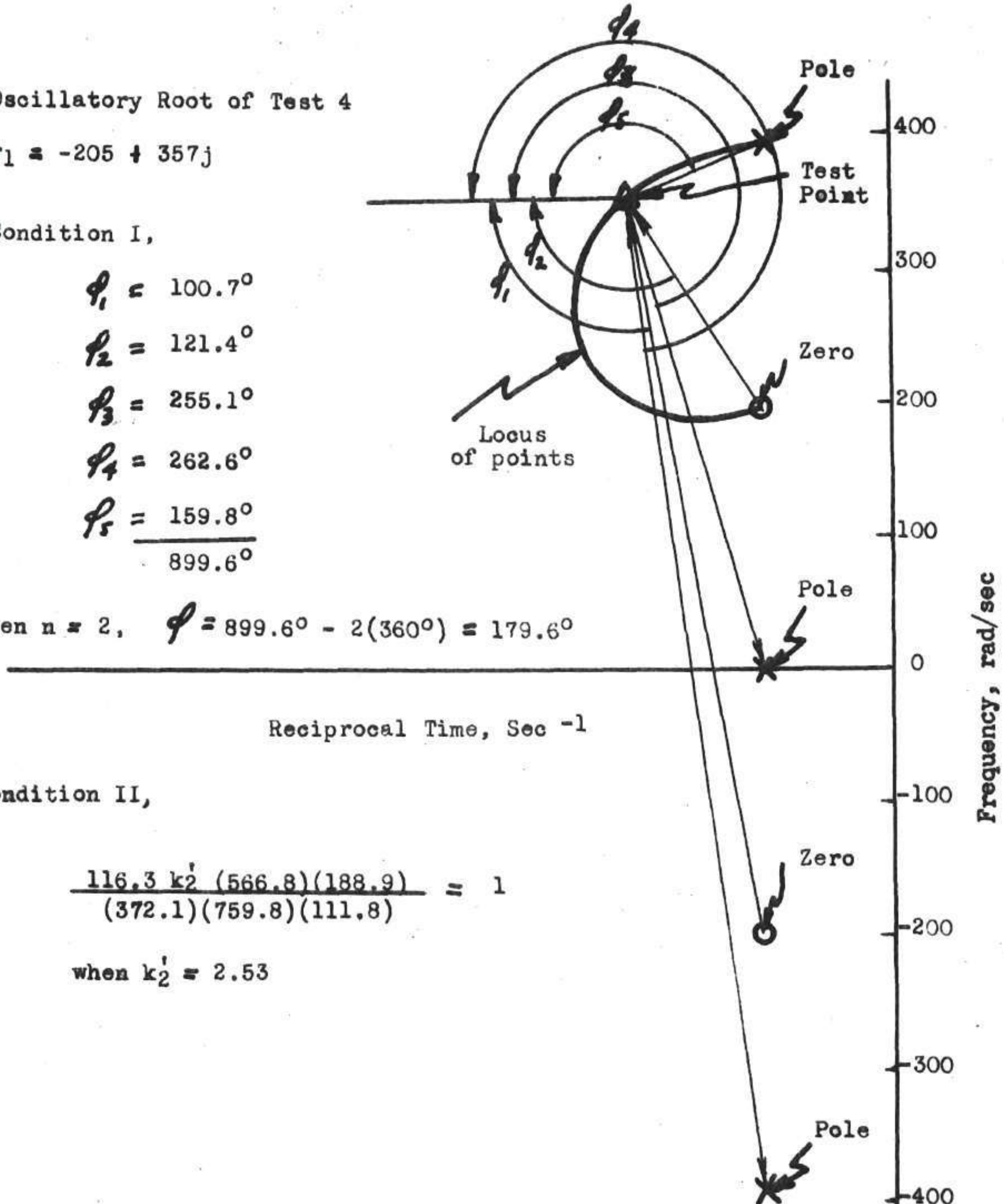


Figure 16, Root Location by Root Locus Method

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